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Dispersion phenomena in transversely isotropic piezo-electric plates with either short or open circuit boundary conditions

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Abstract

The dispersion of small amplitude waves in a transversely isotropic, piezo-electric plate is discussed in respect of both short circuit and open circuit boundary conditions. In both cases the mechanical boundary conditions are taken as traction-free. In both cases, symmetric and anti-symmetric dispersion relations are derived, with long and short wave approximations then established, giving phase speed, and frequency, as functions of scaled wave number. It is shown that some particularly novel features occur within the vicinity of the associated cut-off frequencies. In particular, it is established that for some families the cut-off frequencies depend only on elastic terms, with others depending both on electrical and elastic terms. In each case, the appropriate asymptotic form of displacement is established. This reveals that for motion close to some frequencies, one of the scaled displacements is an order of magnitude larger than the electric potential, however for motion close to other frequencies the opposite situation arises. This information may have applications for the development and design of sensing and actuating devices. The paper also provides the necessary asymptotic framework for the derivation of asymptotically approximate models to fully elucidate the dynamic response of such plates near these resonance frequencies.

Keywords: Dispersion; Piezo-electric; Boundary conditions

1. Introduction

The peculiar electro-mechanical coupling characteristics of piezo-electric materials have, over the past decade or so, been widely exploited in respect of sensing devices, actuators, resonators and various smart structures. In particular, acoustic waves in piezo-electric media are currently being used in a wide range of sensor fields, including physical sensing, chemical sensing and bio-sensing (see Hoummady et al., 1997). The acoustic wave (AW) family of devices includes the surface acoustic wave (SAW), the shear horizontal

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surface acoustic wave (SH SAW), the shear horizontal acoustic plate mode (SH APM), the flexural plate wave (FPW) or Lamb wave mode and thickness shear mode (TSM) devices. Wave propagation in piezo-electric structures therefore continues to attract considerable attention.

Recently, Jin et al. (2002) investigated Lamb wave propagation in a metallic semi-infinite medium covered with a piezo-electric layer. Wang and Varadan (2002) also recently investigated SH waves propagating in piezo-electric layered structures, while Yang and Shue (2001) presented a theoretical and experimental study of leaky Lamb wave (LLW) propagation in a piezo-electric plate loaded by a dielectric/conductive fluid, the latter providing useful information for the development of liquid-based chemical microsensors. Liu et al. (2002a,b) analyzed the propagation of symmetric and anti-symmetric Lamb waves in piezo-electric plates with biasing electric fields. Their study indicated that for a relatively large ratio of plate thickness to wavelength, the maximum fractional velocity change in PZT-5H plates can be up to 0.1%. A further interesting effect, related to a negative biasing electric field, is a possible increase of the electro-mechanical coupling coefficient, one of the most important parameters within the design of piezo-electric sensing devices. Lamb waves present a large sensitivity to mass loading, with in particular the zero order anti-symmetrical mode, when in contact with liquid, having a small attenuation (see for example Laurent et al., 2000).

Wave propagation in elastic plates have been thoroughly studied (see for example Rogerson, 1997 and Pichugin and Rogerson, 2001 and references therein). Rogerson and Kossovich (1999) studied the two-dimensional (plane strain) dispersion relation for a transversely isotropic elastic plate. They derived approximate representations of dispersion relations in the vicinity of their cut-off frequencies, as well as in the short wave high frequency regime. In this paper, we study Lamb waves in a transversely isotropic, piezo-electric plate. Following asymptotic analysis procedures employed in Rogerson and Kossovich's paper, long and short wave approximations for phase speed, and frequency, as functions of wave numbers, together with estimations of the relative order of non-dimensional displacement and electric potential, are derived. These results provide an in-depth insight of Lamb wave propagation in a piezo-electric plate. Moreover, the analysis will be helpful for future studies of leaky Lamb wave problems, which many sensing devices are based on. In particular, the relative asymptotic orders of mechanical displacements and electric potential are also established. This reveals that within the vicinity of some families of cut-off frequencies the scaled displacement in the thickness direction is much larger than scaled electric potential. However within the vicinity of other families, the opposite situation is shown to prevail. These properties suggest that a actuator may be more suitable for working within frequencies associated with former case and a sensor within those associated with the latter. The asymptotic results established in this paper also provide the necessary framework for the derivation of asymptotically approximate models to fully elucidate the dynamic response of such plates near these frequencies, see for example Kaplunov et al. (2000) in respect of three-dimensional motion in transversely isotropic elastic plates.

2. Governing equations and the dispersion relations

We consider the problem of harmonic waves propagating in a plate composed of transversely isotropic piezo-electric material. A Cartesian coordinate system is chosen so that the origin is in the mid-plane and the plate occupies the region $-h \leqslant x \leqslant h$, $-\infty \leqslant y \leqslant \infty$, $-\infty \leqslant z \leqslant \infty$ (see Fig. 1).

The polarization direction of the piezo-electric plate is along the z-axis perpendicular to the xy-plane. The general forms of governing equations for a piezo-electric medium can be expressed in terms of displacements and electric potential as

$$c_{ijkl}\frac{\partial^2 u_l}{\partial x_j \partial x_k} + e_{kij}\frac{\partial^2 \varphi}{\partial x_j \partial x_k} = \rho\frac{\partial^2 u_i}{\partial t^2}, \qquad e_{jkl}\frac{\partial^2 u_l}{\partial x_j \partial x_k} - \varepsilon_{jk}\frac{\partial^2 \phi}{\partial x_j \partial x_k} = 0, \ i,j,k,l = 1,2,3, \tag{1}$$

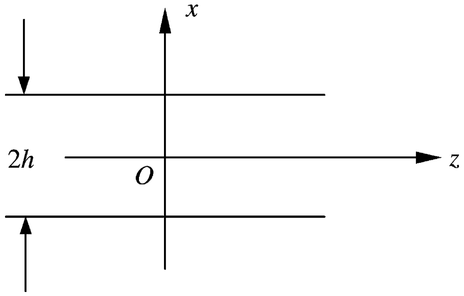

Fig. 1. Geometry of the plate.

where u_l are displacement components, ϕ the electrical potential and ρ the mass density of the medium; also c_{ijkl}, e_{ijk} and ε_{il} are elastic constants, piezo-electric constants and dielectric constants, respectively. In addition, we remark that in the short hand notation used in Eq. (1), x_1, x_2, x_3 are equivalent to x, y, z; also u_1, u_2 and u_3 will also sometimes be replaced by u, v and w. Generally, the stresses σ_{ij} and electric displacements D_i are related to the displacement components and electric potential by

$$\sigma_{ij} = c_{ijkl}u_{k,l} + e_{lij}\phi_{,l}, \qquad D_i = e_{ikl}u_{k,l} - \varepsilon_{il}\phi_{,l}. \tag{2}$$

For the problem of waves propagating along or orthogonal to the preferred direction in a transversely isotropic piezo-electric plate, motion may be decomposed into SH and Lamb wave propagation in the xz-plane. We are only concerned with Lamb waves, so the in-plane displacement component normal to the propagation direction is assumed zero. This is essentially a generalized plane strain problem, in which all quantities are independent of the co-ordinate y. In terms of the standard contracted notation of material properties, the governing equations can be written as

$$\left.\begin{array}{l} c_{11}u_{,xx} + c_{44}u_{,zz} + (c_{13} + c_{44})w_{,xz} + (e_{31} + e_{15})\phi_{,xz} = \rho\ddot{u} \\ (c_{44} + c_{13})u_{,xz} + c_{44}w_{,xx} + c_{33}w_{,zz} + e_{15}\phi_{,xx} + e_{33}\phi_{,zz} = \rho\ddot{w} \\ (e_{15} + e_{31})u_{,xz} + e_{15}w_{,xx} + e_{33}w_{,zz} - \varepsilon_{11}\phi_{,xx} - \varepsilon_{33}\phi_{,zz} = 0 \end{array}\right\}, \tag{3}$$

where a dot indicates differentiation with respect to time.

To facilitate subsequent numerical calculation and asymptotic analysis, it is convenient to make the governing equations non-dimensional, a common approach in dealing with this kind of problem.

Accordingly, we now introduce non-dimensional quantities in the forms

$$\begin{array}{l} \xi = \dfrac{x}{h}, \qquad \eta = \dfrac{z}{h}, \qquad U = \dfrac{u}{h}, \qquad W = \dfrac{w}{h}, \qquad \Phi = \dfrac{e_{33}\phi}{hc_{44}}, \qquad \tau = \dfrac{tv_0}{h}, \\ v_0 = \sqrt{\dfrac{c_{44}}{\rho}}, \qquad \bar{c}_{ij} = \dfrac{c_{ij}}{c_{44}}, \qquad \bar{e}_{ij} = \dfrac{e_{ij}}{e_{33}}, \qquad \bar{\varepsilon}_{ij} = \dfrac{\varepsilon_{ij}c_{44}}{e_{33}^2}. \end{array} \tag{4}$$

The governing equations may now be written in the form

$$\left.\begin{array}{l} \bar{c}_{11}U_{,\xi\xi} + U_{,\eta\eta} + (\bar{c}_{13} + 1)W_{,\xi\eta} + (\bar{e}_{31} + \bar{e}_{15})\Phi_{,\xi\eta} = \ddot{U} \\ (\bar{c}_{11} + 1)U_{,\xi\eta} + W_{,\xi\xi} + \bar{c}_{33}W_{,\eta\eta} + \bar{e}_{15}\Phi_{,\xi\xi} + \Phi_{\eta\eta} = \ddot{W} \\ (\bar{e}_{31} + \bar{e}_{15})U_{,\xi\eta} + \bar{e}_{15}W_{,\xi\xi} + W_{,\eta\eta} - \bar{\varepsilon}_{11}\Phi_{,\xi\xi} - \bar{\varepsilon}_{33}\Phi_{\eta\eta} = 0 \end{array}\right\}. \tag{5}$$

We now seek the solution of Eq. (5) in the form of the harmonic traveling wave

$$(U, W, \Phi) = (\bar{U}(\xi), \bar{W}(\xi), \bar{\Phi}(\xi))\,\mathrm{e}^{\mathrm{i}\bar{k}(\eta - \bar{v}\tau)}, \tag{6}$$

with $\bar{k} = kh$ the non-dimensional wave number and $\bar{v} = v/v_0$ the non-dimensional phase speed. Substituting Eq. (6) into Eq. (5), we have

$$\begin{pmatrix} \bar{c}_{11}\frac{\mathrm{d}^2}{\mathrm{d}\xi^2} - \bar{k}^2 + \bar{k}^2\bar{v}^2 & \mathrm{i}\bar{k}(\bar{c}_{13}+1)\frac{\mathrm{d}}{\mathrm{d}\xi} & \mathrm{i}\bar{k}(\bar{e}_{31}+\bar{e}_{15})\frac{\mathrm{d}}{\mathrm{d}\xi} \\ \mathrm{i}\bar{k}(\bar{c}_{13}+1)\frac{\mathrm{d}}{\mathrm{d}\xi} & \frac{\mathrm{d}^2}{\mathrm{d}\xi^2} - \bar{k}^2\bar{c}_{33} + \bar{k}^2\bar{v}^2 & \bar{e}_{15}\frac{\mathrm{d}^2}{\mathrm{d}\xi^2} - \bar{k}^2 \\ \mathrm{i}\bar{k}(\bar{e}_{31}+\bar{e}_{15})\frac{\mathrm{d}}{\mathrm{d}\xi} & \bar{e}_{15}\frac{\mathrm{d}^2}{\mathrm{d}\xi^2} - \bar{k}^2 & -\bar{\varepsilon}_{11}\frac{\mathrm{d}^2}{\mathrm{d}\xi^2} + \bar{k}^2\bar{\varepsilon}_{33} \end{pmatrix} \begin{Bmatrix} \bar{U} \\ \bar{W} \\ \bar{\Phi} \end{Bmatrix} = 0. \tag{7}$$

It is seen that Eq. (7) has a solution of the form $(\bar{U}, \bar{W}, \bar{\Phi}) = (A, B, C)\mathrm{e}^{\lambda\xi}$, where A, B, C are constants. Substituting this form of solution into Eq. (7) gives

$$\begin{pmatrix} \bar{c}_{11}\lambda^2 - \bar{k}^2 + \bar{k}^2\bar{v}^2 & \mathrm{i}\bar{k}(\bar{c}_{13}+1)\lambda & \mathrm{i}\bar{k}(\bar{e}_{31}+\bar{e}_{15})\lambda \\ \mathrm{i}\bar{k}(\bar{c}_{13}+1)\lambda & \lambda^2 - \bar{k}^2\bar{c}_{33} + \bar{k}^2\bar{v}^2 & \bar{e}_{15}\lambda^2 - \bar{k}^2 \\ \mathrm{i}\bar{k}(\bar{e}_{31}+\bar{e}_{15})\lambda & \bar{e}_{15}\lambda^2 - \bar{k}^2 & -\bar{\varepsilon}_{11}\lambda^2 + \bar{k}^2\bar{\varepsilon}_{33} \end{pmatrix} \begin{Bmatrix} A \\ B \\ C \end{Bmatrix} = 0. \tag{8}$$

In order to obtain a nontrivial solution, the determinant must be zero, thus

$$\begin{vmatrix} \bar{c}_{11}q^2 - 1 + \bar{v}^2 & \mathrm{i}(\bar{c}_{13}+1)q & \mathrm{i}(\bar{e}_{31}+\bar{e}_{15})q \\ \mathrm{i}(\bar{c}_{13}+1)q & q^2 - \bar{c}_{33} + \bar{v}^2 & \bar{e}_{15}q^2 - 1 \\ \mathrm{i}(\bar{e}_{31}+\bar{e}_{15})q & \bar{e}_{15}q^2 - 1 & -\bar{\varepsilon}_{11}q^2 + \bar{\varepsilon}_{33} \end{vmatrix} = 0, \tag{9}$$

in which $\lambda = \bar{k}q$. Eq. (9) provides a cubic equation in q^2, explicitly given by

$$a_3q^6 + a_2q^4 + a_1q^2 + a_0 = 0, \tag{10}$$

where

$$\begin{aligned} a_3 &= \bar{c}_{11}\bar{e}_{15}^2 + \bar{c}_{11}\bar{\varepsilon}_{11}, \\ a_2 &= \bar{c}_{13}^2\bar{\varepsilon}_{11} + 2\bar{c}_{13}\bar{\varepsilon}_{11} + 2\bar{c}_{13}\bar{e}_{15}^2 + \bar{v}^2\bar{e}_{15}^2 + \bar{v}^2\bar{\varepsilon}_{11} - 2\bar{c}_{11}\bar{e}_{15} - \bar{c}_{11}\bar{\varepsilon}_{33} \\ &\quad + \bar{c}_{11}\bar{v}^2\bar{\varepsilon}_{11} - \bar{c}_{11}\bar{c}_{33}\bar{\varepsilon}_{11} + 2\bar{c}_{13}\bar{e}_{31}\bar{e}_{15} - \bar{e}_{31}^2, \\ a_1 &= \bar{c}_{33}\bar{\varepsilon}_{11} - \bar{v}^2\bar{\varepsilon}_{11} - 2\bar{c}_{13}\bar{e}_{31} - 2\bar{c}_{13}\bar{e}_{15} + \bar{e}_{31}^2\bar{c}_{33} - \bar{e}_{31}^2\bar{v}^2 + \bar{e}_{15}^2\bar{c}_{33} \\ &\quad - \bar{v}^2\bar{e}_{15}^2 - \bar{c}_{13}^2\bar{\varepsilon}_{33} - 2\bar{c}_{13}\bar{\varepsilon}_{33} - 2\bar{e}_{15}\bar{v}^2 + \bar{v}^4\bar{\varepsilon}_{11} - \bar{v}^2\bar{\varepsilon}_{33} - \bar{v}^2\bar{c}_{33}\bar{\varepsilon}_{11} \\ &\quad - \bar{c}_{11}\bar{v}^2\bar{\varepsilon}_{33} + \bar{c}_{11}\bar{c}_{33}\bar{\varepsilon}_{33} + 2\bar{e}_{31}\bar{e}_{15}\bar{c}_{33} - 2\bar{e}_{31}\bar{e}_{15}\bar{v}^2 + \bar{c}_{11} - 2\bar{e}_{31}, \\ a_0 &= -1 + \bar{v}^2\bar{c}_{33}\bar{\varepsilon}_{33} + \bar{v}^2\bar{\varepsilon}_{33} - \bar{c}_{33}\bar{\varepsilon}_{33} - \bar{v}^4\bar{\varepsilon}_{33} + \bar{v}^2. \end{aligned} \tag{11}$$

It is remarked that the roots of Eq. (9) may be either real, imaginary or complex. In passing, we note that in the analogous purely elastic plane strain case the analogue of Eq. (10) is only a quadratic in q^2 (see Rogerson and Kossovich, 1999). Assuming that Eq. (10) has three distinct roots, q_1^2, q_2^2 and q_3^2, the complete solutions of Eq. (7) can then be obtained, yielding

$$\begin{Bmatrix} \bar{U} \\ \bar{W} \\ \bar{\Phi} \end{Bmatrix} = \sum_{i=l}^{3} \begin{Bmatrix} A_l\sinh(\bar{k}q_l\xi) \\ B_l\cosh(\bar{k}q_l\xi) \\ C_l\cosh(\bar{k}q_l\xi) \end{Bmatrix} + \sum_{l=1}^{3} \begin{Bmatrix} D_l\cosh(\bar{k}q_l\xi) \\ E_l\sinh(\bar{k}q_l\xi) \\ F_l\sinh(\bar{k}q_l\xi) \end{Bmatrix}, \tag{12}$$

where A_l, B_l, C_l and D_l, E_l, F_l are not independent; but related by $B_l = d_lA_l$, $C_l = f_lA_l$ and $E_l = d_lD_l$, $F_l = f_lD_l$ $(l = 1, 2, 3)$, with d_l and f_l determined by

$$\begin{pmatrix} \mathrm{i}q_l(1+\bar{c}_{13}) & \mathrm{i}q_l(\bar{e}_{15}+\bar{e}_{31}) \\ \bar{e}_{15}q_l^2 - 1 & -\bar{\varepsilon}_{11}q_l^2 + \bar{\varepsilon}_{33} \end{pmatrix} \begin{Bmatrix} d_l \\ f_l \end{Bmatrix} = \begin{pmatrix} -(\bar{c}_{11}q_l^2 - 1 + \bar{v}^2) \\ -\mathrm{i}q_l(\bar{e}_{15}+\bar{e}_{31}) \end{pmatrix}. \tag{13}$$

Eq. (12) may therefore be rewritten as

$$\left\{\begin{matrix}\bar{U}\\ \bar{W}\\ \bar{\Phi}\end{matrix}\right\} = \sum_{l=1}^{3}\begin{pmatrix}\sinh(\bar{k}q_l\xi) & \cosh(\bar{k}q_l\xi)\\ d_l\cosh(\bar{k}q_l\xi) & d_l\sinh(\bar{k}q_l\xi)\\ f_l\cosh(\bar{k}q_l\xi) & f_l\sinh(\bar{k}q_l\xi)\end{pmatrix}\left\{\begin{matrix}A_l\\ D_l\end{matrix}\right\}. \tag{14}$$

The above solutions are only valid for the case of three distinct roots $q_l(l = 1, 2, 3)$. When multiple roots occur, other forms of solution are applicable. However, such forms of solution are omitted here. For our proposed two-dimensional problem, the constitutive relation equation (2) become

$$\begin{aligned}&\sigma_{11} = c_{11}u_{,x} + c_{13}w_{,z} + e_{31}\phi_{,z}, \qquad \sigma_{33} = c_{13}u_{,x} + c_{33}w_{,z} + e_{33}\phi_{,z},\\ &\sigma_{13} = c_{44}(u_{,z} + w_{,x}) + e_{15}\phi_{,x}, \qquad D_1 = e_{15}(u_{,z} + w_{,x}) - \varepsilon_{11}\phi_{,x}, \qquad D_3 = e_{31}u_{,x} + e_{33}w_{,z} - \varepsilon_{33}\phi_{,z}\end{aligned} \tag{15}$$

or in non-dimensional form

$$\begin{aligned}&\bar{\sigma}_{11} = \bar{c}_{11}U_{,\xi} + \bar{c}_{13}W_{,\eta} + \bar{e}_{31}\Phi_{,\eta}, \qquad \bar{\sigma}_{33} = \bar{c}_{13}U_{,\xi} + \bar{c}_{33}W_{,\eta} + \Phi_{,\eta},\\ &\bar{\sigma}_{13} = U_{,\eta} + W_{,\xi} + \bar{e}_{15}\Phi_{,\xi}, \qquad \bar{D}_1 = \bar{e}_{15}(U_{,\eta} + W_{,\xi}) - \bar{\varepsilon}_{11}\Phi_{,\xi}, \qquad \bar{D}_3 = \bar{e}_{31}U_{,\xi} + W_{,\eta} - \bar{\varepsilon}_{33}\Phi_{,\eta},\end{aligned} \tag{16}$$

where $\bar{\sigma}_{ij} = \sigma_{ij}/c_{44}$ and $\bar{D}_i = D_i/e_{33}$. Substitution of Eq. (14) into Eq. (6), and then inserting the resultant expression into Eq. (16) yields

$$\bar{\sigma}_{11} = \left\{\sum_{l=1}^{3}(q_l\bar{c}_{11} + \mathrm{i}d_l\bar{c}_{13} + \mathrm{i}\bar{e}_{31}f_l)[\cosh(\bar{k}q_l\xi)A_l + \sinh(\bar{k}q_l\xi)D_l]\right\}\bar{k}\,\mathrm{e}^{\mathrm{i}\bar{k}(\eta-\bar{v}\tau)}, \tag{17}$$

$$\bar{\sigma}_{13} = \left\{\sum_{l=1}^{3}(\mathrm{i} + q_ld_l + q_lf_l\bar{e}_{15})[\sinh(\bar{k}q_l\xi)A_l + \cosh(\bar{k}q_l\xi)D_l]\right\}\bar{k}\,\mathrm{e}^{\mathrm{i}\bar{k}(\eta-\bar{v}\tau)}, \tag{18}$$

$$\bar{D}_x = \left\{\sum_{l=1}^{3}(\mathrm{i}\bar{e}_{15} + q_ld_l\bar{e}_{15} - q_lf_l\bar{\varepsilon}_{11})[(\sinh(\bar{k}q_l\xi)A_l + \cosh(\bar{k}q_l\xi)D_l]\right\}\bar{k}\,\mathrm{e}^{\mathrm{i}\bar{k}(\eta-\bar{v}\tau)}. \tag{19}$$

The expressions of stresses and electric potential have now been obtained. We are therefore in a position to derive the dispersion relations by imposing appropriate boundary conditions on the upper and lower surfaces.

Case 1: Traction free and electrically grounded

In this case $\sigma_{11}, \sigma_{13}, \phi\,|_{x=\pm h} = 0$ and hereinafter these conditions will be referred to as the short circuit condition. Imposing these boundary conditions leads to dispersion relations for extensional and flexural waves.

(a) Extensional waves

In the case of extensional waves, $\mathrm{Det}\{a_{kl}\} = 0$, where

$$a_{1l} = q_l\bar{c}_{11} + \mathrm{i}d_l\bar{c}_{13} + \mathrm{i}\bar{e}_{31}f_l, \qquad a_{2l} = (\mathrm{i} + q_ld_l + q_lf_l\bar{e}_{15})\tanh(q_l\bar{k}), \qquad a_{3l} = f_l, \tag{20}$$

or written in the following form as

$$H_1\tanh(q_1\bar{k}) + H_2\tanh(q_2\bar{k}) + H_3\tanh(q_3\bar{k}) = 0, \tag{21}$$

in which H_l $(l = 1, 2, 3)$ are constants dependent on material properties, q_l and $\bar{v}$. Here, for brevity, we omit the expressions of H_l. We remark that in Eq. (20) there is no implied summation over repeated suffices.

(b) Flexural waves

In the flexural case, $\mathrm{Det}\{a_{kl}\} = 0$, where

$$a_{1l} = q_l\bar{c}_{11} + \mathrm{i}d_l\bar{c}_{13} + \mathrm{i}\bar{e}_{31}f_l, \qquad a_{2l} = (\mathrm{i} + q_ld_l + q_if_l\bar{e}_{15})\coth(q_l\bar{k}), \qquad a_{3l} = f_l, \tag{22}$$

or written in the following form as

$$H_1\coth(q_1\bar{k}) + H_2\coth(q_2\bar{k}) + H_3\coth(q_3\bar{k}) = 0. \tag{23}$$

Case 2: Traction and electric charge free

These boundary conditions are reasonable when the plate is placed in the air. This type of boundary condition will hereafter referred to as the open circuit condition. The dispersion relations can readily be obtained and they are given below.

(a) Extensional waves

For extensional waves, the dispersion relation is provided by $\mathrm{Det}\{a_{kl}\} = 0$, where

$$\begin{aligned} &a_{1l} = (q_l\bar{c}_{11} + \mathrm{i}d_l\bar{c}_{13} + \mathrm{i}\bar{e}_{31}f_l)\coth(q_l\bar{k}), \qquad a_{2l} = \mathrm{i} + q_ld_l + q_lf_l\bar{e}_{15}, \\ &a_{3l} = \mathrm{i}\bar{e}_{15} + \bar{e}_{15}q_ld_l - \bar{\varepsilon}_{11}q_lf_l, \end{aligned} \tag{24}$$

or written in the following form as

$$G_1\coth(q_1\bar{k}) + G_2\coth(q_2\bar{k}) + G_3\coth(q_3\bar{k}) = 0, \tag{25}$$

in which G_l $(l = 1, 2, 3)$ are constants which depend on material properties, q_l and $\bar{v}$. Here we omit the expressions of G_l for brevity and remark that in (24) there is no implied summation over repeated suffices.

(b) Flexural waves

The flexural dispersion relation is given by $\mathrm{Det}\{a_{kl}\} = 0$, where

$$\begin{aligned} &a_{1l} = (q_l\bar{c}_{11} + \mathrm{i}d_l\bar{c}_{13} + \mathrm{i}\bar{e}_{31}f_l)\tanh(q_l\bar{k}), \qquad a_{2l} = \mathrm{i} + q_ld_l + q_lf_l\bar{e}_{15}, \\ &a_{3l} = \mathrm{i}\bar{e}_{15} + \bar{e}_{15}q_ld_l - \bar{\varepsilon}_{11}q_lf_l, \end{aligned} \tag{26}$$

or written in the following form as

$$G_1\tanh(q_1\bar{k}) + G_2\tanh(q_2\bar{k}) + G_3\tanh(q_3\bar{k}) = 0. \tag{27}$$

2.1. Numerical results

The dispersion relations (21), (23), (25) and (27) are all transcendental equations, giving scaled phase speed $\bar{v}$ as an implicit function of scaled wave number $\bar{k}$. Since the dispersion relations are continuous functions, and either real or purely imaginary, we can use the bisection method to solve the dispersion relations. However, there are a few multiple roots for which bisection method does not work. For these special cases, we employ the modified Newton method (see for example Van Loan, 1997). It should also be noted that $\sinh(\cdot)$ and $\cosh(\cdot)$ should be employed for all numerical calculations, instead of $\tanh(\cdot)$ and $\coth(\cdot)$, as $\tanh(\cdot)$ and $\coth(\cdot)$ may become infinite at some points.

Numerical calculation is made for PZT-4 piezo-electric ceramics in this paper. The material properties of PZT-4 are taken as those reported by Wang and Noda (2002): $c_{11} = 13.9 \times 10^{10}$ N/m^2, $c_{13} = 7.43 \times 10^{10}$ N/m^2, $c_{33} = 11.3 \times 10^{10}$ N/m^2, $c_{44} = 2.56 \times 10^{10}$ N/m^2; the piezo-electric constants are $e_{31} = -6.98$ C/m^2, $e_{33} = 13.84$ C/m^2, $e_{15} = 13.44$ C/m^2; the dielectric constants are $\varepsilon_{11} = 60.0 \times 10^{-10}$ C/V m, $\varepsilon_{33} = 57.4 \times 10^{-10}$ C/V m.

Some numerical results are presented in Figs. 2–9, with Figs. 2–5 showing the variation of scaled phase velocity with scaled wave number and Figs. 6–9 scaled frequency ($\bar{\omega} = \bar{v}\bar{k}$) against scaled wave number. For the material parameters employed, the two scaled body wave speeds, obtained from Eq. (10) with $q = 0$, are given by $\bar{v}_1 = 1$ and $\bar{v}_2 = 2.39115$ and numerical results indicate that the short wave limit of all harmonics is the lower of these two. This indicates that for the harmonics, as $\bar{k} \to \infty$, $\bar{v} \to 1$ and $|q| \to 0$. From these figures it may also be seen that fundamental mode branches approach a wave speed slightly lower that this body wave speed limit as $\bar{k} \to \infty$. This is in fact the associated surface wave speed. These type of modes are characterised by the roots of the secular equation (10) being either all real, or one real root accompanied by a complex conjugate pair. A consequence of this is that the phase speed associated with each dispersion relation is a single-valued function of wave number. A wave front traveling at the higher of the two body

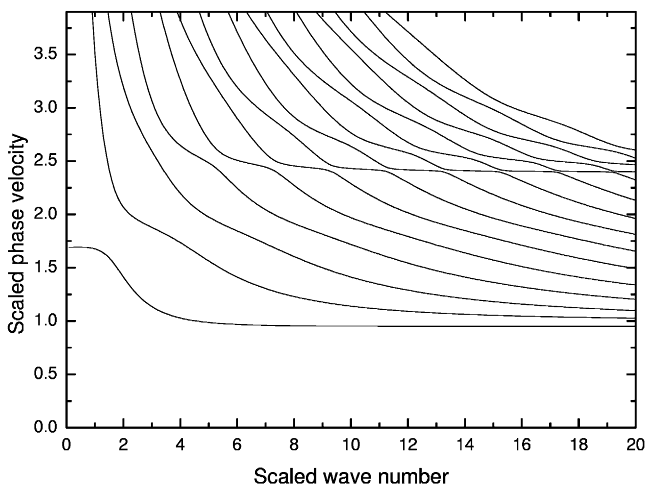

Fig. 2. Numerical solutions, scaled phase velocity against scaled wave number, of extensional dispersion relation; short circuit condition.

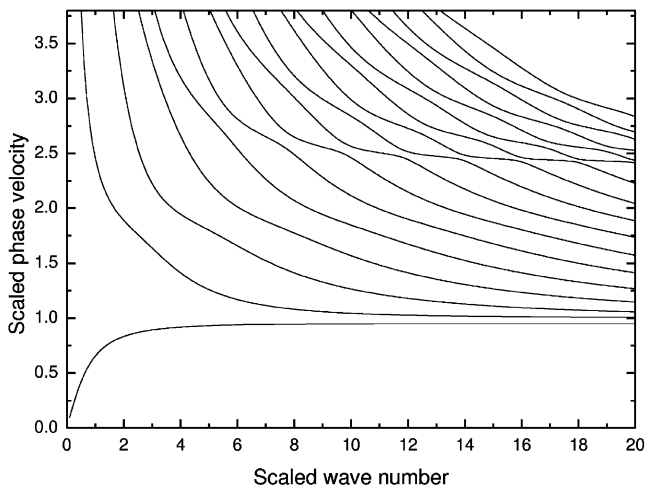

Fig. 3. Numerical solutions, scaled phase velocity against scaled wave number, of flexural dispersion relation; short circuit condition.

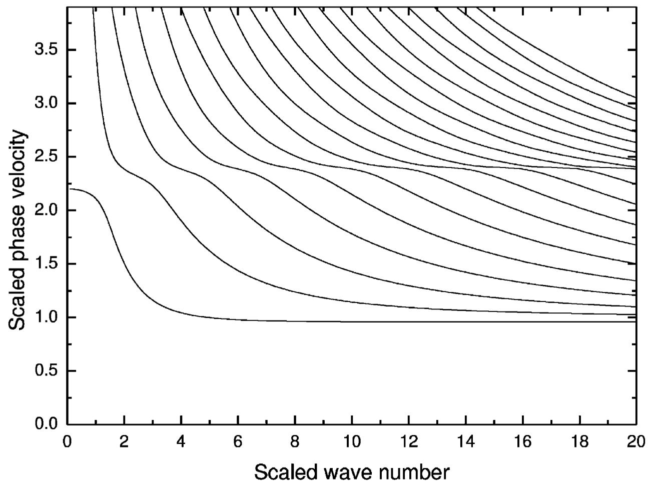

Fig. 4. Numerical solution, scaled phase velocity against scaled wave number, of the extensional dispersion relation; open circuit condition.

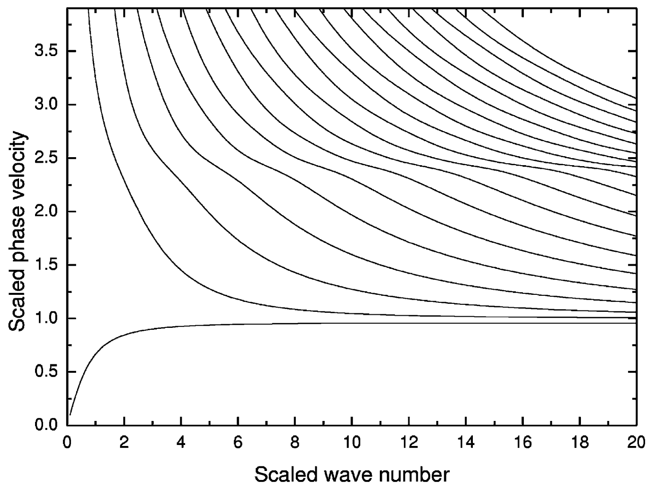

Fig. 5. Numerical solution, scaled phase velocity against scaled wave number, of the flexural dispersion relation; open circuit condition.

wave speeds is also observed to be formed through the flattening of dispersion relation curves, indicating turning points of the associated group velocity curves. This flattening is more pronounced in the case of extensional waves. Unlike the case of an elastic plate, oscillations of dispersion relation curves do not occur for these parameters. Usually oscillation occurs when the secular equation (10) has two equal roots as $\bar{k} \to \infty$. For PZT-4, Eq. (10) two equal roots will occur only when the scaled phase velocity $\bar{v} = 0.69636$, which lies below the surface waves speeds. In the long wave limit we note that as in the elastic case, it is only the fundamental mode associated with extensional motion which has finite non-zero long wave phase speed limit. In respect of all harmonics, we remark that as $\bar{k} \to 0$, $\bar{v} \gg 1$. In Figs. 6,8,9, corresponding plots for scaled frequency against scaled wave number are presented. These clearly show that the frequency of the fundamental modes tend to zero in the long wave limit, with the cut-off frequencies of the harmonics all non-zero. In the next section these numerical results will be compared with some approximations derived by

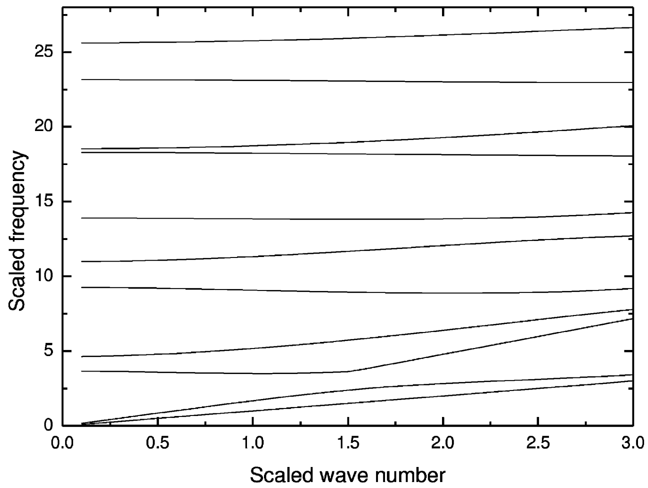

Fig. 6. Numerical solution, scaled frequency against scaled wave number, of the extensional dispersion relation; short circuit condition.

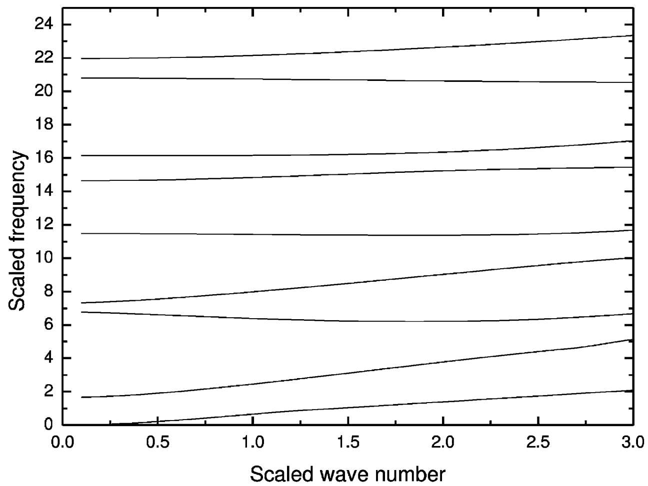

Fig. 7. Numerical solution, scaled frequency against scaled wave number, of the flexural dispersion relation; short circuit condition.

asymptotic analysis. Although there does not appear to be much qualitative difference between extension/flexural motion or the open/short circuit boundary conditions, we shall see that careful analysis reveals some subtle, but non the less important, differences.

3. Analysis of the dispersion relations

3.1. Approximation near cut-off frequencies

We now seek explicit expressions for frequencies, as functions of wave number, in the long wave high frequency regime, that is in the vicinity of the non-zero cut-off frequencies. It is seen from the last section

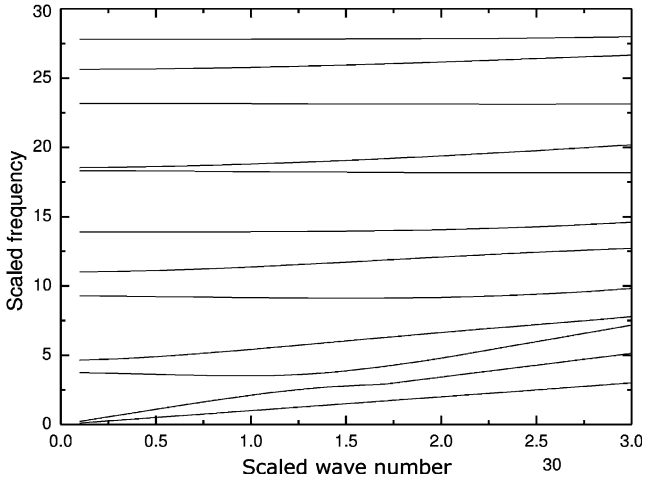

Fig. 8. Numerical solution, scaled frequency against scaled wave number, of the extensional dispersion relation; open circuit condition.

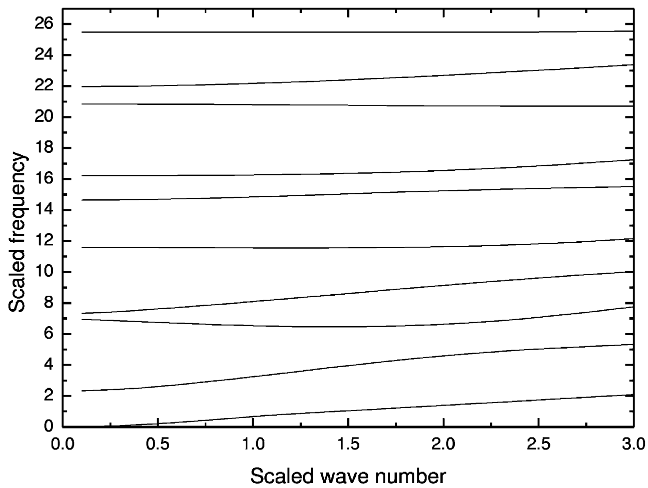

Fig. 9. Numerical solution, scaled frequency against scaled wave number, of the flexural dispersion relation; open circuit condition.

that for this type of motion $\bar{v} \to \infty$ as $\bar{k} \to 0$. Analysis of the relative orders of the coefficients of the secular equation (10) reveals that $q_1^2 + q_2^2 + q_3^2$ is $\mathrm{O}(\bar{v}^2)$, that $q_1^2 q_2^2 + q_1^2 q_3^2 + q_2^2 q_3^2$ is $\mathrm{O}(\bar{v}^4)$ and that $q_1^2 q_2^2 q_3^2$ is $\mathrm{O}(\bar{v}^4)$. Accordingly we deduce that two roots are of the order $\mathrm{O}(\bar{v}^2)$, with another root of order of $\mathrm{O}(1)$. Specifically, approximations for q_1^2, q_2^2 and q_3^2 are given by

$$q_1^2 = Q_1^{(1)} + Q_1^{(2)}\bar{v}^{-2} + \mathrm{O}(\bar{v}^{-4}), \qquad q_m^2 = -Q_m^{(1)}\bar{v}^2 + Q_m^{(2)} + \mathrm{O}(\bar{v}^{-2}), \tag{28}$$

with here, and throughout the paper, $m = 2, 3$ and where

$$Q_1^{(1)} = \frac{\bar{\varepsilon}_{33}}{\bar{\varepsilon}_{11}}, \qquad Q_1^{(2)} = \frac{\bar{\varepsilon}_{11}^2 - \bar{\varepsilon}_{33}(\bar{\varepsilon}_{11}\bar{e}_{31} + \bar{\varepsilon}_{11}\bar{e}_{15}^2 + 2\bar{\varepsilon}_{11}\bar{e}_{15} + 2\bar{\varepsilon}_{11}\bar{e}_{15}\bar{e}_{31} - \bar{\varepsilon}_{33}\bar{e}_{15}^2)}{\bar{\varepsilon}_{11}^3},$$

$$Q_2^{(2)} = \frac{\bar{c}_{11}(\bar{e}_{15}^2 + 2\bar{e}_{31}\bar{e}_{15} + \bar{e}_{31}^2 + \bar{\varepsilon}_{11}) + \bar{c}_{13}(2\bar{e}_{15}^2 + 2\bar{e}_{31}\bar{e}_{15} + \bar{c}_{13}\bar{\varepsilon}_{11} + 2\bar{\varepsilon}_{11}) - \bar{e}_{31}^2}{\bar{c}_{11}(\bar{e}_{15}^2 + \bar{\varepsilon}_{11} - \bar{c}_{11}\bar{\varepsilon}_{11}))},$$

$$Q_2^{(1)} = \frac{1}{\bar{c}_{11}}, \qquad Q_3^{(1)} = \frac{\bar{\varepsilon}_{11}}{\bar{e}_{15}^2 + \bar{\varepsilon}_{11}},$$

$$\begin{aligned}\bar{\varepsilon}_{11}(\bar{e}_{15}^2 + \bar{\varepsilon}_{11})(\bar{e}_{15}^2 + \bar{\varepsilon}_{11} - \bar{c}_{11}\bar{\varepsilon}_{11})Q_3^{(2)} = {}& \bar{\varepsilon}_{33}\bar{e}_{15}^4 - 2\bar{e}_{31}\bar{e}_{15}\bar{\varepsilon}_{11}^2 - \bar{e}_{15}^2\bar{c}_{33}\bar{\varepsilon}_{11}^2 - 2\bar{\varepsilon}_{11}\bar{e}_{31}\bar{e}_{15}^3 - \bar{\varepsilon}_{11}\bar{e}_{31}^2\bar{e}_{15}^2 - 2\bar{\varepsilon}_{11}^2\bar{c}_{13}\bar{e}_{15}^2 \\ &+ \bar{\varepsilon}_{11}^3\bar{c}_{11}\bar{c}_{33} + 2\bar{\varepsilon}_{11}^2\bar{c}_{11}\bar{e}_{15} + \bar{\varepsilon}_{33}\bar{e}_{15}^2\bar{\varepsilon}_{11} - \bar{\varepsilon}_{11}\bar{c}_{11}\bar{\varepsilon}_{33}\bar{e}_{15}^2 - \bar{\varepsilon}_{11}^3 \\ &- 2\bar{\varepsilon}_{11}^2\bar{c}_{13}\bar{e}_{31}\bar{e}_{15} - \bar{\varepsilon}_{11}e_{15}^4 - 2\bar{e}_{15}^2\bar{\varepsilon}_{11}^2 - \bar{c}_{33}\bar{\varepsilon}_{11}^3 - 2\bar{\varepsilon}_{11}\bar{e}_{15}^3 - 2\bar{e}_{15}\bar{\varepsilon}_{11}^2 \\ &- \bar{\varepsilon}_{11}^3\bar{c}_{13}^2 - 2\bar{\varepsilon}_{11}^3\bar{c}_{13}.\end{aligned}$$

We will also need expansions for q_1, q_2 and q_3, these are given by

$$q_1 = \sqrt{Q_1^{(1)}} + \frac{Q_1^{(2)}}{2\sqrt{Q_1^{(1)}}}\bar{v}^{-2} + \mathrm{O}(\bar{v}^{-4}), \qquad q_m = \mathrm{i}\left\{\sqrt{Q_m^{(1)}}\bar{v} - \frac{Q_m^{(2)}}{2\sqrt{Q_m^{(1)}}}\bar{v}^{-1} + \mathrm{O}(\bar{v}^{-3})\right\}. \tag{29}$$

From Eq. (29) we are also now able to derive $\bar{v}$ in terms of q, thus

$$\bar{v}_m^2 = -q_m^2/Q_m^{(1)} + Q_m^{(2)}/Q_m^{(1)} + \mathrm{O}(q_m^{-2}). \tag{30}$$

We may also substitute Eq. (29) into Eq. (13) to reveal that

$$\begin{aligned}&d_1 = S_1 + \mathrm{O}(\bar{v}^{-2}), \qquad f_1 = T_1\bar{v}^2 + \mathrm{O}(1), \qquad d_2 = S_2/\bar{v} + \mathrm{O}(\bar{v}^{-3}),\\ &f_2 = T_2/\bar{v} + \mathrm{O}(\bar{v}^{-3}), \qquad d_3 = S_3\bar{v} + \mathrm{O}(\bar{v}^{-1}), \qquad f_3 = T_3\bar{v} + \mathrm{O}(\bar{v}^{-1}),\end{aligned} \tag{31}$$

where

$$\begin{aligned}S_1 &= -\frac{\mathrm{i}(\bar{\varepsilon}_{33}\bar{e}_{31}^2 + \bar{\varepsilon}_{33}\bar{e}_{15}^2 + 2\bar{e}_{15}\bar{e}_{31}\bar{\varepsilon}_{33} - 2Q_1^{(2)}\bar{\varepsilon}_{11}^2)\sqrt{\bar{\varepsilon}_{11}}}{(\bar{\varepsilon}_{33}\bar{e}_{15}^2 + \bar{e}_{15}\bar{\varepsilon}_{33}\bar{e}_{31} - \bar{e}_{15}\bar{\varepsilon}_{11} - \bar{e}_{31}\bar{\varepsilon}_{11})\sqrt{\bar{\varepsilon}_{33}}},\\ T_1 &= \frac{\mathrm{i}\sqrt{\bar{\varepsilon}_{11}}}{(\bar{e}_{15} + \bar{e}_{31})\sqrt{\bar{\varepsilon}_{33}}},\\ S_2 &= \frac{(\bar{\varepsilon}_{11}\bar{c}_{11}Q_2^{(2)} - 2\bar{e}_{31}\bar{e}_{15} - \bar{e}_{15}^2 - \bar{e}_{31}^2 - \bar{\varepsilon}_{11})\sqrt{\bar{c}_{11}}}{(\bar{\varepsilon}_{11} + \bar{e}_{31}\bar{e}_{15} + \bar{e}_{15}^2 + \bar{c}_{13}\bar{\varepsilon}_{11})},\\ T_2 &= \frac{(\bar{e}_{31} + \bar{e}_{15}\bar{c}_{13} + \bar{e}_{31}\bar{c}_{13} + \bar{e}_{15}\bar{c}_{11}Q_2^{(2)})\sqrt{\bar{c}_{11}}}{(\bar{\varepsilon}_{11} + \bar{e}_{31}\bar{e}_{15} + \bar{e}_{15}^2 + \bar{c}_{13}\bar{\varepsilon}_{11})},\\ S_3 &= \frac{\bar{\varepsilon}_{11}(Q_3^{(1)}\bar{c}_{11} - 1)}{(\bar{\varepsilon}_{11} + \bar{e}_{31}\bar{e}_{15} + \bar{e}_{15}^2 + \bar{c}_{13}\bar{\varepsilon}_{11})\sqrt{Q_3^{(1)}}},\\ T_3 &= \frac{\bar{e}_{15}(Q_3^{(1)}\bar{c}_{11} - 1)}{(\bar{\varepsilon}_{11} + \bar{e}_{31}\bar{e}_{15} + \bar{e}_{15}^2 + \bar{c}_{13}\bar{\varepsilon}_{11})\sqrt{Q_3^{(1)}}}.\end{aligned} \tag{32}$$

Furthermore, using Eqs. (29) and (31) it is possible to establish that

$$H_1 = \mathrm{i}h_1^{(1)}\bar{v}^4 + \mathrm{O}(\bar{v}^2), \qquad H_2 = h_2^{(1)}\bar{v}^3 + \mathrm{O}(\bar{v}), \qquad H_3 = h_3^{(1)}\bar{v}^5 + \mathrm{O}(\bar{v}^3), \tag{33}$$

where

$$
\begin{aligned}
h_1^{(1)} &= -\sqrt{Q_2^{(1)}}\bar{e}_{15}T_1\sqrt{Q_1^{(1)}}T_3\bar{c}_{11},\\
h_2^{(1)} &= \sqrt{Q_2^{(1)}}T_1\left(\sqrt{Q_3^{(1)}}\bar{c}_{11}(\bar{e}_{15}T_2+S_2)+S_3\bar{c}_{13}(S_2+\bar{e}_{15}T_2)\right)+T_1\left(S_3\bar{c}_{13}+\sqrt{Q_3^{(1)}}\bar{c}_{11}\right),\\
h_3^{(1)} &= -\sqrt{Q_3^{(1)}}\sqrt{Q_2^{(1)}}\bar{e}_{15}T_1T_3\bar{c}_{11}-\sqrt{Q_3^{(1)}}\sqrt{Q_2^{(1)}}T_1S_3\bar{c}_{11}.
\end{aligned}
\tag{34}
$$

Similarly, approximations for G_i $(i = 1, 2, 3)$ are obtained in the form

$$
G_1 = \mathrm{i}g_1^{(1)}\bar{v}^4 + \mathrm{O}(\bar{v}^2), \qquad G_2 = g_2^{(1)}\bar{v}^5 + \mathrm{O}(\bar{v}^3), \qquad G_3 = g_3^{(1)}\bar{v}^3 + \mathrm{O}(\bar{v}), \tag{35}
$$

where

$$
\begin{aligned}
g_1^{(1)} &= \sqrt{Q_3^{(1)}}\bar{e}_{31}T_1\bar{e}_{15}^2T_3\left(\bar{e}_{15}^2+\varepsilon_{11}+\sqrt{Q_2^{(1)}}\bar{e}_{15}^2S_2\right)\\
&\quad+\sqrt{Q_2^{(1)}}\sqrt{Q_3^{(1)}}\bar{e}_{31}T_1S_2\varepsilon_{11}T_3\left(S_2\varepsilon_{11}T_3-\bar{e}_{15}^2T_2S_3-S_3\varepsilon_{11}T_2\right),\\
g_2^{(1)} &= \sqrt{Q_2^{(1)}}\sqrt{Q_1^{(1)}}\sqrt{Q_3^{(1)}}T_1\bar{c}_{11}S_3(\varepsilon_{11}+\bar{e}_{15}^2),\\
g_3^{(1)} &= -\sqrt{Q_1^{(1)}}T_1\left(\bar{e}_{31}\bar{e}_{15}^2T_3+\bar{e}_{31}\varepsilon_{11}T_3+\sqrt{Q_3^{(1)}}\varepsilon_{11}\bar{c}_{11}+\sqrt{Q_2^{(1)}}\bar{e}_{15}^2S_3\bar{c}_{13}S_2\right.\\
&\quad\left.+\sqrt{Q_2^{(1)}}\varepsilon_{11}S_3\bar{c}_{13}S_2+\sqrt{Q_2^{(1)}}\bar{e}_{31}\bar{e}_{15}^2T_3S_2+\varepsilon_{11}S_3\bar{c}_{13}\right).
\end{aligned}
\tag{36}
$$

3.1.1. Short circuit condition

Extensional waves: For extensional waves, the leading order approximation of the dispersion relation (21) is given by

$$
\mathrm{i}h_1^{(1)}\bar{v}\tanh(\bar{k}q_1) + h_2^{(1)}\tanh(\bar{k}q_2) + h_3^{(1)}\bar{v}^2\tanh(\bar{k}q_3) \sim 0. \tag{37}
$$

Eq. (37) indicates that the dispersion relation may be asymptotically balanced only if $\tanh(\bar{k}q_2) \sim \mathrm{O}(\bar{v}^2)$ or $\tanh(\bar{k}q_3) \sim \mathrm{O}(\bar{v}^{-2})$. In the first case, we can deduce that

$$
\bar{k}q_2 = \mathrm{i}\left(\left(n-\frac{1}{2}\right)\pi + \Gamma_1^{(1)}\bar{k}^2 + \mathrm{O}(\bar{k}^4)\right), \tag{38}
$$

where $\Gamma_1^{(1)}$ is a constant, which is determined by substituting Eq. (38) into Eq. (37) and equating terms with like powers of $\bar{k}$, to obtain

$$
\Gamma_1^{(1)} = \frac{h_2^{(1)}Q_2^{(1)}}{h_3^{(1)}\left(n-\frac{1}{2}\right)^2\pi^2\tan\left(\sqrt{Q_3^{(1)}}(n-\frac{1}{2})\pi/\sqrt{Q_2^{(1)}}\right)}. \tag{39}
$$

By making use of Eq. (30), an approximation of scaled frequency $\bar{\omega} = \bar{v}\bar{k}$ is derived, namely

$$
\bar{\omega}^2 = \left(n-\frac{1}{2}\right)^2\frac{\pi^2}{Q_2^{(1)}} + \left(Q_2^{(2)} + 2\left(n-\frac{1}{2}\right)\pi\Gamma_1^{(1)}\right)\frac{\bar{k}^2}{Q_2^{(1)}} + \mathrm{O}(\bar{k}^4), \tag{40}
$$

in which the square root of the first term $(n-\frac{1}{2})^2\pi^2/Q_2^{(1)}$ defines the associated cut-off frequency. In the second case, we deduce that

$$
\bar{k}q_3 = \mathrm{i}\left\{n\pi + \Gamma_2^{(1)}\bar{k}^2 + \mathrm{O}(\bar{k}^4)\right\}, \tag{41}
$$

enabling us to infer that

$$\Gamma_2^{(1)} = -\frac{\left(\sqrt{Q_1^{(1)}}\sqrt{Q_3^{(1)}}h_1^{(1)}n\pi + Q_3^{(1)}h_2\tan\left(\sqrt{Q_2^{(1)}}n\pi/\sqrt{Q_3^{(1)}}\right)\right)}{n^2\pi^2 h_3}. \tag{42}$$

Finally, we obtain the analogue of (40) in the form

$$\bar{\varpi}^2 = n^2\pi^2/Q_3^{(1)} + (Q_3^{(2)} + 2n\pi\Gamma_2^{(1)})\bar{k}^2/Q_3^{(1)} + \mathrm{O}(\bar{k}^4). \tag{43}$$

Flexural waves: For flexural waves, with the short circuit condition, the leading order approximation of the dispersion relation (23) is expressible as

$$\mathrm{i}h_1^{(1)}\bar{v}\coth(\bar{k}q_1) + h_2^{(1)}\coth(\bar{k}q_2) + h_3^{(1)}\bar{v}^2\coth(\bar{k}q_3) \sim 0. \tag{44}$$

From Eq. (44) we notice that in order for the dispersion relation to be asymptotically balanced, either $\coth(\bar{k}q_2) \sim \mathrm{O}(\bar{v}^2)$ or $\coth(\bar{k}q_3) \sim \mathrm{O}(1)$. In the first case, we infer that

$$\bar{k}q_2 = \mathrm{i}\left\{n\pi + \Gamma_3^{(1)}\bar{k}^2 + \mathrm{O}(\bar{k}^4)\right\}, \tag{45}$$

which after substituting into Eq. (44) reveals that

$$\Gamma_3^{(1)} = -\frac{h_2^{(1)}Q_2^{(1)}\sqrt{Q_1^{(1)}}}{n\pi h_1^{(1)}\sqrt{Q_2^{(1)}} + n^2\pi^2 h_3^{(1)}\cot\left(n\pi\sqrt{Q_3^{(1)}}/\sqrt{Q_2^{(1)}}\right)}. \tag{46}$$

By using Eq. (30), the approximate expression for scaled frequency is obtainable, thus

$$\bar{\varpi}^2 = n^2\pi^2/Q_2^{(1)} + (Q_2^{(2)} + 2n\pi\Gamma_3^{(1)})\bar{k}^2/Q_2^{(1)} + \mathrm{O}(\bar{k}^4). \tag{47}$$

$$\bar{k}q_3 = \mathrm{i}\left\{\Gamma_4^{(1)} + \Gamma_5^{(1)}\bar{k}^2 + \mathrm{O}(\bar{k}^4)\right\}, \tag{48}$$

with $\Gamma_4^{(1)}$ a the root of the following transcendental equation

$$\Gamma_4^{(1)}\cot(\Gamma_4^{(1)}) = -\frac{h_1^{(1)}\sqrt{Q_3^{(1)}}}{h_3^{(1)}\sqrt{Q_1^{(1)}}}. \tag{49}$$

Substituting Eq. (48) into Eq. (44), and making use of Eq. (49), yields

$$\Gamma_5^{(1)} = \frac{h_1^{(1)}\left(2(Q_1^{(1)})^2\sqrt{Q_3^{(1)}}(\Gamma_4^{(1)})^2 - 3Q_1^{(2)}\sqrt{Q_3^{(1)}}Q_3^{(1)}\right) + 6h_2^{(1)}Q_3^{(1)}\sqrt{Q_1^{(1)}}Q_1^{(1)}\Gamma_4^{(1)}\mathscr{C}_4}{6Q_1^{(1)}\sqrt{Q_1^{(1)}}h_3^{(1)}(\Gamma_4^{(1)})^3(1+\cot^2(\Gamma_4^{(1)}))}, \tag{50}$$

in which

$$\mathscr{C}_4 = \cot\left(\Gamma_4^{(1)}\sqrt{Q_2^{(1)}}/\sqrt{Q_3^{(1)}}\right). \tag{51}$$

Finally, we obtain the approximation

$$\bar{\varpi}^2 = (\Gamma_4^{(1)})^2/Q_3^{(1)} + (2\Gamma_4^{(1)}\Gamma_5^{(1)} + Q_3^{(2)})\bar{k}^2/Q_3^{(1)} + \mathrm{O}(\bar{k}^4). \tag{52}$$

3.1.2. Open circuit condition

Extensional waves: For extensional waves, the leading order approximation of the dispersion relation equation (25) is given by

$$\mathrm{i}g_1^{(1)}\bar{v}\coth(\bar{k}q_1) + g_2^{(1)}\bar{v}^2\coth(\bar{k}q_2) + g_3^{(1)}\coth(\bar{k}q_3) \sim 0. \tag{53}$$

Eq. (53) indicates that the dispersion relation may be asymptotically balanced only if $\coth(\bar{k}q_2) \sim \mathrm{O}(1)$ or $\coth(\bar{k}q_3) \sim \mathrm{O}(\bar{v}^2)$. In the first case, we deduce that

$$\bar{k}q_2 = \mathrm{i}\left\{\Gamma_1^{(2)} + \Gamma_2^{(2)}\bar{k}^2 + \mathrm{O}(\bar{k}^4)\right\}, \tag{54}$$

where $\Gamma_1^{(2)}$ is determined by the following transcendental equation

$$\Gamma_1^{(2)}\cot(\Gamma_1^{(2)}) = -\frac{g_1^{(1)}\sqrt{Q_2^{(1)}}}{g_2^{(1)}\sqrt{Q_1^{(1)}}}, \tag{55}$$

with $\Gamma_2^{(2)}$ given by

$$\Gamma_2^{(2)} = \frac{g_1^{(1)}\left(2(\Gamma_1^{(2)})^2(Q_1^{(1)})^2\sqrt{Q_2^{(1)}} - 3Q_1^{(2)}Q_2^{(1)}\sqrt{Q_2^{(1)}}\right) + 6g_3^{(1)}\Gamma_1^{(2)}Q_1^{(1)}Q_2^{(1)}\sqrt{Q_1^{(1)}}\mathscr{C}_2}{6g_2^{(1)}(\Gamma_1^{(2)})^3Q_1^{(1)}\sqrt{Q_1^{(1)}}(1+\cot^2(\Gamma_1^{(2)}))}, \tag{56}$$

where

$$\mathscr{C}_2 = \cot\left(\Gamma_1^{(2)}\sqrt{Q_3^{(1)}}/\sqrt{Q_2^{(1)}}\right).$$

The corresponding approximate expression for scaled frequency may now be derived as

$$\bar{\varpi}^2 = (\Gamma_1^{(2)})^2/Q_2^{(1)} + (Q_2^{(2)} + 2\Gamma_1^{(2)}\Gamma_2^{(2)})\bar{k}^2/Q_2^{(1)} + \mathrm{O}(\bar{k}^4). \tag{57}$$

In the second case, it is readily established that

$$\bar{k}q_3 = \mathrm{i}\left\{n\pi + \Gamma_3^{(2)}\bar{k}^2 + \mathrm{O}(\bar{k}^4)\right\}. \tag{58}$$

Substituting Eq. (58) into Eq. (53) gives

$$\Gamma_3^{(2)} = -\frac{\sqrt{Q_1^{(1)}}Q_3^{(1)}g_3^{(1)}}{n\pi\sqrt{Q_3^{(1)}}g_1^{(1)} + n^2\pi^2\sqrt{Q_1^{(1)}}g_2^{(1)}}, \tag{59}$$

with the approximate expression for scaled frequency given by

$$\bar{\varpi}^2 = n^2\pi^2/Q_3^{(1)} + (Q_3^{(2)} + 2n\pi\Gamma_3^{(2)})\bar{k}^2/Q_3^{(1)} + \mathrm{O}(\bar{k}^4). \tag{60}$$

Flexural waves: For flexural waves, with the open circuit condition, the leading order approximation of the dispersion relation (27) is

$$\mathrm{i}g_1^{(1)}\bar{v}\tanh(\bar{k}q_1) + g_2^{(1)}\bar{v}^2\tanh(\bar{k}q_2) + g_3^{(1)}\tanh(\bar{k}q_3) \sim 0. \tag{61}$$

From Eq. (61), note that the only possibilities are either $\tanh(\bar{k}q_2) \sim \mathrm{O}(\bar{v}^{-2})$ or $\tanh(\bar{k}q_3) \sim \mathrm{O}(\bar{v}^2)$. In the first case, we deduce that

$$\bar{k}q_2 = \mathrm{i}\left\{n\pi + \Gamma_4^{(2)}\bar{k}^2 + \mathrm{O}(\bar{k}^4)\right\}, \tag{62}$$

with $\Gamma_4^{(2)}$ determined by substituting Eq. (62) into Eq. (61) and equating like powers of $\bar{k}$, to establish that

$$\Gamma_4^{(2)} = -\frac{n\pi g_1^{(1)}\sqrt{Q_1^{(1)}Q_2^{(1)}} + g_3^{(1)}Q_2^{(1)}\tan\left(n\pi\sqrt{Q_3^{(1)}/Q_2^{(1)}}\right)}{n^2\pi^2 g_2^{(1)}}. \tag{63}$$

By use of Eq. (30), the approximate expression of scaled frequency is found to be

$$\bar{\omega}^2 = n^2\pi^2/Q_2^{(1)} + (Q_2^{(2)} + 2n\pi\Gamma_4^{(2)})\bar{k}^2/Q_2^{(1)} + \mathrm{O}(\bar{k}^4). \tag{64}$$

For the second case, the analogue of (62) is given by

$$\bar{k}q_3 = \mathrm{i}\left\{\left(n-\frac{1}{2}\right)\pi + \Gamma_5^{(2)}\bar{k}^2 + \mathrm{O}(\bar{k}^4)\right\}, \tag{65}$$

where, by use of previously employed procedures, $\Gamma_5^{(2)}$ is obtained as

$$\Gamma_5^{(2)} = \frac{g_3^{(1)}Q_3^{(1)}}{g_2^{(1)}\left(n-\frac{1}{2}\right)^2\pi^2\tan\left(\left(n-\frac{1}{2}\right)\pi\sqrt{Q_2^{(1)}/Q_3^{(1)}}\right)}, \tag{66}$$

with the corresponding frequency approximation given by

$$\bar{\omega}^2 = \left(n-\frac{1}{2}\right)^2\frac{\pi^2}{Q_3^{(1)}} + \left(Q_3^{(2)} + 2\left(n-\frac{1}{2}\right)\pi\Gamma_5^{(2)}\right)\frac{\bar{k}^2}{Q_3^{(1)}} + \mathrm{O}(\bar{k}^4). \tag{67}$$

In Figs. 10–13, comparison of numerical results with asymptotic solutions for scaled frequencies in the vicinity of cut-off frequencies is made. These figures show good agreement over a relatively large wave number region. There are a few points related to the long wave high frequency cases which are worthy of some further comment. We first remark that in two cases see Eqs. (49), and (55), the cut-off frequencies are

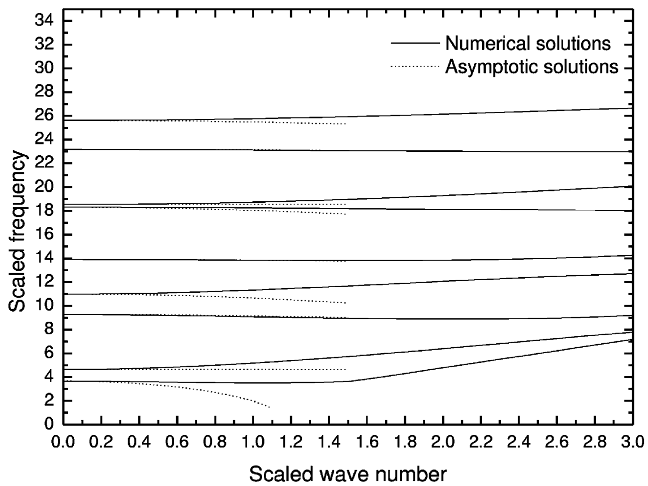

Fig. 10. Comparison of asymptotic and numerical solutions in the vicinity of the cut-off frequencies; extensional case, short circuit condition.

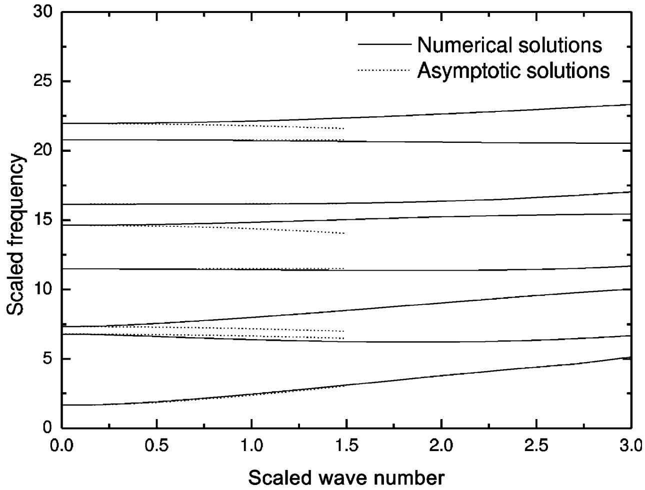

Fig. 11. Comparison of asymptotic and numerical solutions in the vicinity of the cut-off frequencies; flexural case, short circuit condition.

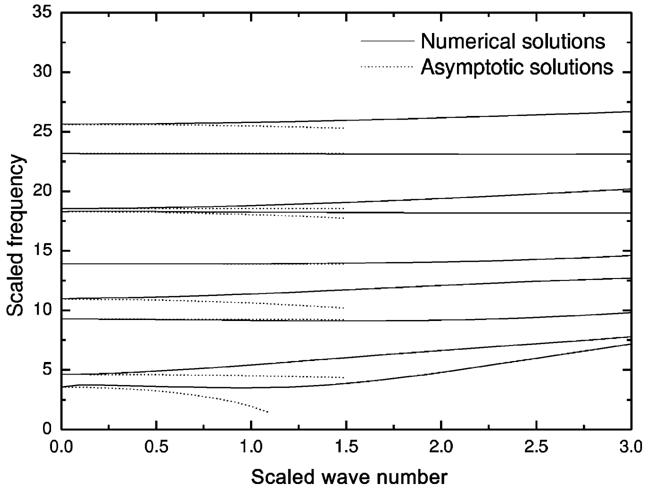

Fig. 12. Comparison of asymptotic and numerical solutions in the vicinity of the cut-off frequencies; extensional case, open circuit condition.

defined by the roots of transcendental equations, rather than explicit expressions. Although this is not the case in the corresponding elastic problem, i.e. the traction free case, it has recently been shown to be a feature of dispersion in incompressible elastic plates with fixed faces, see Kaplunov and Nolde (2002) and Nolde and Rogerson (2002). However, in these studies it is only a feature of symmetric motion; whereas in the present case it is a feature both in the extensional case, with open circuit conditions, and in the flexural case, with the short circuit conditions. A further noteworthy point is that some of the cut-off frequencies involve only elastic material parameters, see Eqs. (40), (47) and (64), with all other involving a combination of both electric and elastic parameters, see Eqs. (43), (49), (55), (60) and (67) and note the scales introduced in equation (4).

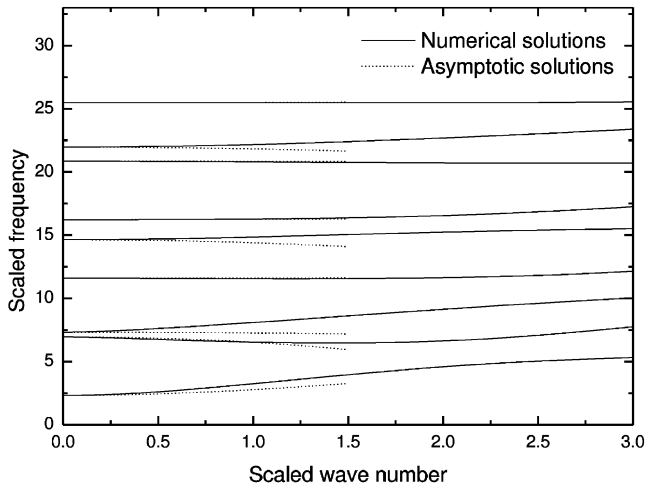

Fig. 13. Comparison of asymptotic and numerical solutions in the vicinity of the cut-off frequencies; flexural case, open circuit condition.

3.2. Short wave high frequency approximation

In the short wave case, we remark that $\bar{k} \gg 1$. In this case $\bar{v} > 1$ and $\bar{v} \to 1$ from above, accordingly we have $q_1 = \mathrm{i}\hat{q}_1$, with $\hat{q}_1 > 0$ and tending to zero in the short wave limit, indicating that

$$\bar{v}^2 = 1 + Q_1\hat{q}_1^2 + \mathrm{O}(\hat{q}_1^4), \tag{68}$$

where

$$Q_1 = (\bar{c}_{33}\bar{\varepsilon}_{11} - 2\bar{c}_{13}\bar{e}_{31} - 2\bar{c}_{13}\bar{e}_{15} + \bar{e}_{31}^2\bar{c}_{33} + \bar{e}_{15}^2\bar{c}_{33} - \bar{c}_{13}^2\bar{\varepsilon}_{33} - 2\bar{c}_{13}\bar{\varepsilon}_{33} + \bar{c}_{11}\bar{c}_{33}\bar{\varepsilon}_{33} + 2\bar{e}_{31}\bar{e}_{15}\bar{c}_{33} + \bar{c}_{11} - 2\bar{e}_{31}) \\ /(1 + \bar{\varepsilon}_{33}\bar{c}_{33} - \bar{\varepsilon}_{33}). \tag{69}$$

Corresponding approximations for q_2^2 and q_3^2 are given by

$$q_2^2 = Q_2 + \mathrm{O}(\hat{q}_1^2), \qquad q_3^2 = Q_3 + \mathrm{O}(\hat{q}_1^2), \tag{70}$$

where

$$Q_2 = \left(\frac{-2b_2 + \sqrt{b_2^2 - 4b_1b_3}}{2b_3}\right), \qquad Q_3 = \left(\frac{-2b_2 - \sqrt{b_2^2 - 4b_1b_3}}{2b_3}\right) \tag{71}$$

and

$$\begin{aligned} b_1 &= -2\bar{c}_{13}\bar{e}_{31} - 2\bar{c}_{13}\bar{e}_{15} + \bar{e}_{31}^2\bar{c}_{33} - \bar{e}_{31}^2 + \bar{e}_{15}^2\bar{c}_{33} - \bar{e}_{15}^2 - \bar{c}_{13}^2\bar{\varepsilon}_{33} - 2\bar{c}_{13}\bar{\varepsilon}_{33} - 2\bar{e}_{15} - \bar{\varepsilon}_{33} - \bar{c}_{11}\bar{\varepsilon}_{33} + \bar{c}_{11}\bar{c}_{33}\bar{\varepsilon}_{33} \\ &\quad + 2\bar{e}_{31}\bar{e}_{15}\bar{c}_{33} - 2\bar{e}_{31}\bar{e}_{15} + \bar{c}_{11} - 2\bar{e}_{31}, \\ b_2 &= \bar{c}_{13}^2\bar{\varepsilon}_{11} + 2\bar{c}_{13}\bar{\varepsilon}_{11} + 2\bar{c}_{13}\bar{e}_{15}^2 + \bar{e}_{15}^2 + \bar{\varepsilon}_{11} - 2\bar{c}_{11}\bar{e}_{15} - \bar{c}_{11}\bar{\varepsilon}_{33} + \bar{c}_{11}\bar{\varepsilon}_{11} - \bar{c}_{11}\bar{c}_{33}\bar{\varepsilon}_{11} + 2\bar{c}_{13}\bar{e}_{31}\bar{e}_{15} - \bar{e}_{31}^2, \\ b_3 &= \bar{c}_{11}\bar{e}_{15}^2 + \bar{c}_{11}\bar{\varepsilon}_{11}. \end{aligned} \tag{72}$$

Substituting Eqs. (68) and (70) and $q_1 = \mathrm{i}\hat{q}_1$ into Eq. (13), we obtain

$$\begin{aligned} d_1 &= M_1\hat{q}_1 + \mathrm{O}(\hat{q}_1^3), \qquad d_2 = \mathrm{i}\left\{M_2 + \mathrm{O}(\hat{q}_1^2)\right\}, \qquad d_3 = \mathrm{i}\left\{M_3 + \mathrm{O}(\hat{q}_1^2)\right\}, \\ f_1 &= N_1\hat{q}_1 + \mathrm{O}(\hat{q}_1^3), \qquad f_2 = \mathrm{i}\left\{N_2 + \mathrm{O}(\hat{q}_1^2)\right\}, \qquad f_3 = \mathrm{i}\left\{N_3 + \mathrm{O}(\hat{q}_1^2)\right\}, \end{aligned} \tag{73}$$

where

$$M_1 = \frac{-2\bar{e}_{15}\bar{e}_{31} - \bar{c}_{11}\bar{\varepsilon}_{33} + Q_1\bar{\varepsilon}_{33} - \bar{e}_{31}^2 - \bar{e}_{15}^2}{\bar{\varepsilon}_{33}\bar{c}_{13} + \bar{\varepsilon}_{33} + \bar{e}_{31} + \bar{e}_{15}},$$

$$N_1 = \frac{\bar{e}_{31}\bar{c}_{13} - \bar{c}_{11} + Q_1 + \bar{e}_{15} + \bar{e}_{31} + \bar{e}_{15}\bar{c}_{13}}{(\bar{\varepsilon}_{33}\bar{c}_{13} + \bar{\varepsilon}_{33} + \bar{e}_{31} + \bar{e}_{15})},$$

$$M_2 = \frac{\sqrt{Q_2}(Q_2\bar{\varepsilon}_{11}\bar{c}_{11} - \bar{e}_{15}^2 - 2\bar{e}_{15}\bar{e}_{31} - \bar{e}_{31}^2 - \bar{c}_{11}\bar{\varepsilon}_{33})}{(\bar{c}_{13}\bar{\varepsilon}_{11}Q_2 - \bar{\varepsilon}_{33}\bar{c}_{13} + \bar{\varepsilon}_{11}Q_2 - \bar{\varepsilon}_{33} + \bar{e}_{31}\bar{e}_{15}Q_2 - \bar{e}_{31} + \bar{e}_{15}^2Q_2 - \bar{e}_{15})},$$

$$N_2 = \frac{\sqrt{Q_2}(\bar{e}_{31}\bar{c}_{13} + \bar{e}_{15}\bar{c}_{13} + \bar{e}_{31} + \bar{e}_{15} + Q_2\bar{c}_{11}\bar{e}_{15} - \bar{c}_{11})}{(\bar{c}_{13}\bar{\varepsilon}_{11}Q_2 - \bar{\varepsilon}_{33}\bar{c}_{13} + \bar{\varepsilon}_{11}Q_2 - \bar{\varepsilon}_{33} + \bar{e}_{31}\bar{e}_{15}Q_2 - \bar{e}_{31} + \bar{e}_{15}^2Q_2 - \bar{e}_{15})},$$

$$M_3 = \frac{\sqrt{Q_3}(Q_3\bar{\varepsilon}_{11}\bar{c}_{11} - \bar{e}_{15}^2 - 2\bar{e}_{15}\bar{e}_{31} - \bar{e}_{31}^2 - \bar{c}_{11}\bar{\varepsilon}_{33})}{(\bar{c}_{13}\bar{\varepsilon}_{11}Q_3 - \bar{\varepsilon}_{33}\bar{c}_{13} + \bar{\varepsilon}_{11}Q_3 - \bar{\varepsilon}_{33} + \bar{e}_{31}\bar{e}_{15}Q_3 - \bar{e}_{31} + \bar{e}_{15}^2Q_3 - \bar{e}_{15})},$$

$$N_3 = \frac{\sqrt{Q_3}(\bar{e}_{31}\bar{c}_{13} + \bar{e}_{15}\bar{c}_{13} + \bar{e}_{31} + \bar{e}_{15} + Q_3\bar{c}_{11}\bar{e}_{15} - \bar{c}_{11})}{(\bar{c}_{13}\bar{\varepsilon}_{11}Q_3 - \bar{\varepsilon}_{33}\bar{c}_{13} + \bar{\varepsilon}_{11}Q_3 - \bar{\varepsilon}_{33} + \bar{e}_{31}\bar{e}_{15}Q_3 - \bar{e}_{31} + \bar{e}_{15}^2Q_3 - \bar{e}_{15})}.$$

Furthermore, using Eqs. (70) and (73), as well setting $q_1 = \mathrm{i}\hat{q}_1$, reveals that

$$H_1 = h_1^{(2)} + \mathrm{O}(\hat{q}_1^2), \qquad H_2 = \mathrm{i}\left\{h_2^{(2)}\hat{q}_1 + \mathrm{O}(\hat{q}_1^3)\right\}, \qquad H_3 = \mathrm{i}\left\{h_3^{(2)}\hat{q}_1 + \mathrm{O}(\hat{q}_1^3)\right\}, \tag{74}$$

where

$$\begin{aligned} h_1^{(2)} &= N_3\bar{c}_{11}\sqrt{Q_2} + N_2M_3\bar{c}_{13} - N_2c_{11}\sqrt{Q_3} - N_3M_2\bar{c}_{13}, \\ h_2^{(2)} &= N_1M_3\bar{c}_{13} - \sqrt{Q_2}\bar{c}_{11}N_3M_2 - N_3\bar{c}_{11} - \sqrt{Q_2}\bar{c}_{11}N_3\bar{e}_{15}N_2 - N_1\bar{c}_{11}\sqrt{Q_3} - \sqrt{Q_2}M_1\bar{c}_{13}N_3M_2 \\ &\quad - \sqrt{Q_2}N_1\bar{c}_{11}\sqrt{Q_3}M_2 - M_1\bar{c}_{13}N_3 + \sqrt{Q_2}N_1M_3\bar{c}_{13}\bar{e}_{15}N_2 - \sqrt{Q_2}N_1\bar{c}_{11}\sqrt{Q_3}\bar{e}_{15}N_2 \\ &\quad - \sqrt{Q_2}M_1\bar{c}_{13}N_3\bar{e}_{15}N_2 + \sqrt{Q_2}N_1M_3\bar{c}_{13}M_2, \\ h_3^{(2)} &= N_1\bar{c}_{11}\sqrt{Q_2Q_3}\bar{e}_{15}N_3 - \sqrt{Q_3}N_1M_2\bar{c}_{13}M_3 + M_1\bar{c}_{13}N_2 + \sqrt{Q_3}M_1\bar{c}_{13}N_2M_3 \\ &\quad + \sqrt{Q_3}M_1\bar{c}_{13}N_2\bar{e}_{15}N_3 + N_1\bar{c}_{11}\sqrt{Q_2Q_3}M_3 - \sqrt{Q_3}N_1M_2\bar{c}_{13}\bar{e}_{15}N_3 - N_1M_2\bar{c}_{13} + \sqrt{Q_2}N_1\bar{c}_{11} \\ &\quad + \sqrt{Q_3}\bar{c}_{11}N_2\bar{e}_{15}N_3 + N_2\bar{c}_{11} + \sqrt{Q_3}\bar{c}_{11}N_2M_3. \end{aligned} \tag{75}$$

Similar approximations for G_i $(i = 1, 2, 3)$ are also obtainable, namely

$$G_1 = \mathrm{i}\left\{g_1^{(2)}\hat{q}_1 + \mathrm{O}(\hat{q}_1^3)\right\}, \qquad G_2 = g_2^{(2)} + \mathrm{O}(\hat{q}_1^2), \qquad G_3 = g_3^{(2)} + \mathrm{O}(\hat{q}_1^2), \tag{76}$$

where

$$g_1^{(2)} = \sqrt{Q_3Q_2}\{\bar{e}_{31}N_1\bar{e}_{15}^2N_3M_2 + M_1\bar{c}_{13}M_2\bar{\varepsilon}_{11}N_3 - M_1\bar{c}_{13}M_3\bar{\varepsilon}_{11}N_2 - \bar{e}_{31}N_1M_3\bar{\varepsilon}_{11}N_2 - M_1\bar{c}_{13}\bar{e}_{15}^2N_2M_3 - \bar{c}_{11}M_3\bar{\varepsilon}_{11}N_2 - \bar{c}_{11}\bar{e}_{15}^2N_2M_3 + \bar{c}_{11}M_2\bar{\varepsilon}_{11}N_3 + \bar{e}_{31}N_1M_2\bar{\varepsilon}_{11}N_3 + \bar{c}_{11}\bar{e}_{15}^2N_3M_2 + M_1\bar{c}_{13}\bar{e}_{15}^2N_3M_2 - \bar{e}_{31}N_1\bar{e}_{15}^2N_2M_3\} - \sqrt{Q_2}\{\bar{e}_{31}N_1\bar{e}_{15}^2N_2 + M_1\bar{c}_{13}\bar{e}_{15}^2N_2 + M_1\bar{c}_{13}\bar{\varepsilon}_{11}N_2 + \bar{c}_{11}\bar{e}_{15}^2N_2 + \bar{e}_{31}N_1\bar{\varepsilon}_{11}N_2 + \bar{c}_{11}\bar{\varepsilon}_{11}N_2\} + \sqrt{Q_3}\{M_1\bar{c}_{13}\bar{\varepsilon}_{11}N_3 + \bar{c}_{11}\bar{\varepsilon}_{11}N_3 + \bar{c}_{11}\bar{e}_{15}^2N_3 + M_1\bar{c}_{13}\bar{e}_{15}^2N_3 + \bar{e}_{31}N_1\bar{\varepsilon}_{11}N_3 + \bar{e}_{31}N_1\bar{e}_{15}^2N_3\},$$

$$g_2^{(2)} = -\bar{e}_{15}^2\bar{c}_{11}\sqrt{Q_2}\sqrt{Q_3}N_3 - \bar{c}_{11}\sqrt{Q_2}\bar{\varepsilon}_{11}\sqrt{Q_3}N_3 + M_2\bar{c}_{13}\bar{\varepsilon}_{11}\sqrt{Q_3}N_3 + \bar{e}_{31}N_2\bar{\varepsilon}_{11}\sqrt{Q_3}N_3 + \bar{e}_{15}^2M_2\bar{c}_{13}\sqrt{Q_3}N_3 + \bar{e}_{15}^2\bar{e}_{31}N_2\sqrt{Q_3}N_3,$$

$$g_3^{(2)} = \bar{e}_{15}^2\bar{c}_{11}\sqrt{Q_3}\sqrt{Q_2}N_2 - M_3\bar{c}_{13}\bar{\varepsilon}_{11}\sqrt{Q_2}N_2 - \bar{e}_{15}^2M_3\bar{c}_{13}\sqrt{Q_2}N_2 - \bar{e}_{15}^2\bar{e}_{31}N_3\sqrt{Q_2}N_2 + \bar{c}_{11}\sqrt{Q_3}\bar{\varepsilon}_{11}\sqrt{Q_2}N_2 - \bar{e}_{31}N_3\bar{\varepsilon}_{11}\sqrt{Q_2}N_2.$$

3.2.1. Short circuit condition

Extensional waves: For extensional waves, the dispersion relation (21) has the following approximate form:

$$h_1^{(2)}\tan(\bar{k}\hat{q}_1) + h_2^{(2)}\hat{q}_1 + h_3^{(2)}\hat{q}_1 \sim 0. \tag{77}$$

The above equation implies that for $\bar{k} \gg 1$

$$\tan(\bar{k}\hat{q}_1) \sim \mathrm{O}(\hat{q}_1), \tag{78}$$

from which we infer that

$$\bar{k}\hat{q}_1 = n\pi + \Lambda_1^{(1)}\bar{k}^{-1} + \mathrm{O}(\bar{k}^{-2}). \tag{79}$$

It is now possible to substitute Eq. (78) into Eq. (77) to establish that

$$\Lambda_1^{(1)} = -(h_2^{(2)} + h_3^{(2)})\frac{n\pi}{h_1^{(2)}}. \tag{80}$$

Substituting Eq. (79) into Eq. (68), we obtain the approximate expression of scaled phase velocity in the following form

$$\bar{v}^2 = 1 + n^2\pi^2Q_1\bar{k}^{-2} + 2n\pi Q_1\Lambda_1^{(1)}\bar{k}^{-3} + \mathrm{O}(\bar{k}^{-4}). \tag{81}$$

Flexural waves: For flexural harmonics in the short wave limit, the dispersion Eq. (23) can be represented in the following approximate form

$$-h_1^{(2)}\cot(\bar{k}\hat{q}_1) + h_2^{(2)}\hat{q}_1 + h_3^{(2)}\hat{q}_1 \sim 0. \tag{82}$$

From the above equation, it is seen that for $\bar{k} \gg 1$

$$\cot(\bar{k}\hat{q}_1) \sim \mathrm{O}(\hat{q}_1), \tag{83}$$

enabling us to deduce that

$$\bar{k}\hat{q}_1 = \left(n - \frac{1}{2}\right)\pi + \Lambda_2^{(1)}\bar{k}^{-1} + \mathrm{O}(\bar{k}^{-2}) \tag{84}$$

and thus

$$\Lambda_2^{(1)} = -\left(h_2^{(2)} + h_3^{(2)}\right)\left(n - \frac{1}{2}\right)\frac{\pi}{h_1^{(2)}}, \tag{85}$$

finally arriving at

$$\bar{v}^2 = 1 + \left(n - \frac{1}{2}\right)^2 \pi^2 Q_1 \bar{k}^{-2} + 2\left(n - \frac{1}{2}\right)\pi Q_1 \Lambda_2^{(1)} \bar{k}^{-3} + \mathrm{O}(\bar{k}^{-4}). \tag{86}$$

3.2.2. Open circuit condition

Extensional waves: For extensional waves, the dispersion relation takes the following approximate form

$$g_1^{(2)}\hat{q}_1 \cot(\bar{k}\hat{q}_1) + g_2^{(2)} + g_3^{(2)} \sim 0, \tag{87}$$

indicating that for $\bar{k} \gg 1$

$$\cot(\bar{k}\hat{q}_1) \sim \mathrm{O}(\hat{q}_1^{-1}). \tag{88}$$

We may therefore infer that

$$\bar{k}\hat{q}_1 = n\pi + \Lambda_1^{(2)}\bar{k}^{-1} + \mathrm{O}(\bar{k}^{-2}), \tag{89}$$

which after substituting Eq. (89) into Eq. (87) reveals that

$$\Lambda_1^{(2)} = -n\pi g_1^{(2)}/(g_2^{(2)} + g_3^{(2)}). \tag{90}$$

Making use of Eq. (68) an approximate for the scaled phase velocity is obtainable, namely

$$\bar{v}^2 = 1 + n^2\pi^2 Q_1 \bar{k}^{-2} + 2n\pi Q_1 \Lambda_1^{(2)} \bar{k}^{-3} + \mathrm{O}(\bar{k}^{-4}) \tag{91}$$

Flexural waves: For the harmonics associated with flexural modes, the dispersion relation is of the following approximate form

$$-g_1^{(2)}\hat{q}_1 \tan(\bar{k}\hat{q}_1) + g_2^{(2)} + g_3^{(2)} \sim 0. \tag{92}$$

From this equation we infer that for $\bar{k} \gg 1$

$$\tan(\bar{k}\hat{q}_1) \sim \mathrm{O}(\hat{q}_1^{-1}), \tag{93}$$

from which we deduce that

$$\bar{k}\hat{q}_1 = \left(n - \frac{1}{2}\right)\pi + \Lambda_2^{(2)}\bar{k}^{-1} + \mathrm{O}(\bar{k}^{-2}), \tag{94}$$

whence following similar lines of thought to previous, we get

$$\Lambda_2^{(2)} = -\left(n - \frac{1}{2}\right)\pi g_1^{(2)}/(g_2^{(2)} + g_3^{(2)}). \tag{95}$$

Finally, an approximation for the scaled phase velocity for this case is obtained in the form

$$\bar{v}^2 = 1 + \left(n - \frac{1}{2}\right)^2 \pi^2 Q_1 \bar{k}^{-2} + 2\left(n - \frac{1}{2}\right)\pi Q_1 \Lambda_2^{(2)} \bar{k}^{-3} + \mathrm{O}(\bar{k}^{-4}). \tag{96}$$

In Figs. 14–17 various short wave approximations of the harmonics are shown against numerical solutions. These indicate a good agreement over a surprisingly large wave number region.

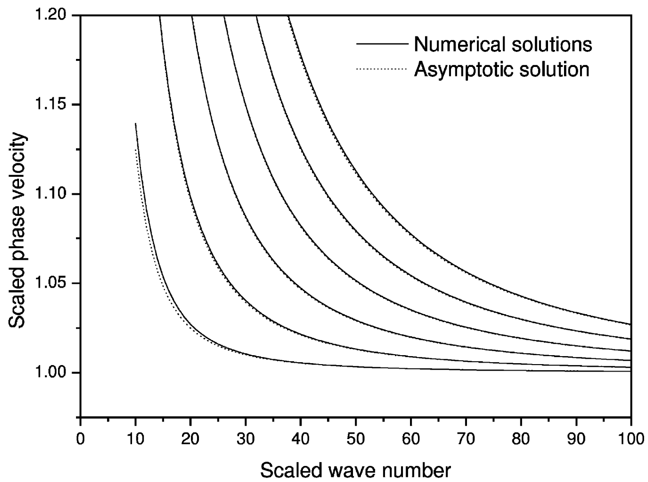

Fig. 14. Comparison of asymptotic and numerical solutions in the short wave regime; extensional case, short circuit condition.

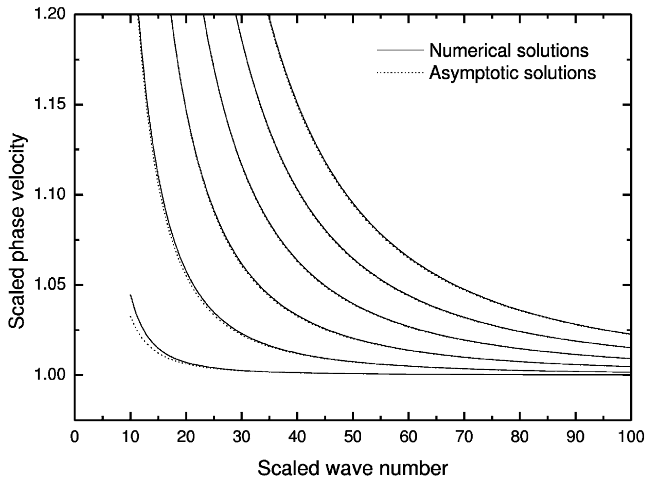

Fig. 15. Comparison of asymptotic and numerical solutions in the short wave regime; flexural case, short circuit condition.

3.3. Short wavelength limit for the fundamental mode

3.3.1. Short circuit condition

In the short wave limit of the fundamental mode, $\bar{k} \to \infty$ and $\bar{v} \to 1$ from below, with at least one of q_1, q_2 and q_3 real and the other two either real or forming a complex conjugate pair. Both dispersion relations for the short circuit conditions, see Eqs. (21) and (23), become

$$H_1 + H_2 + H_3 = 0. \tag{97}$$

3.3.2. Open circuit condition

In the open circuit case, the analogue of (97) is given by

$$G_1 + G_2 + G_3 = 0. \tag{98}$$

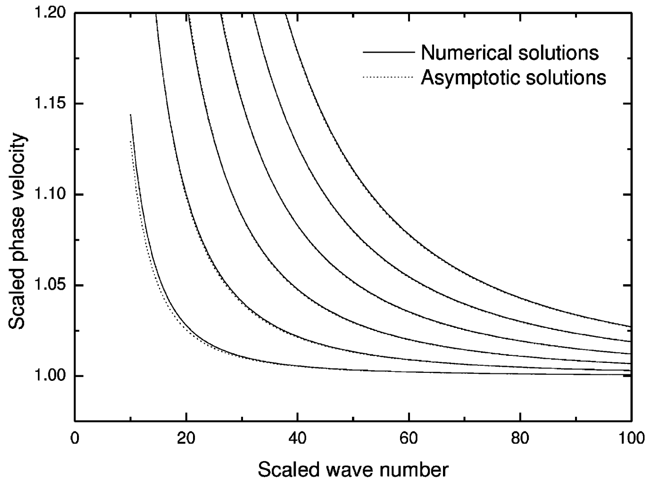

Fig. 16. Comparison of asymptotic and numerical solutions in the short wave regime; extensional case, open circuit condition.

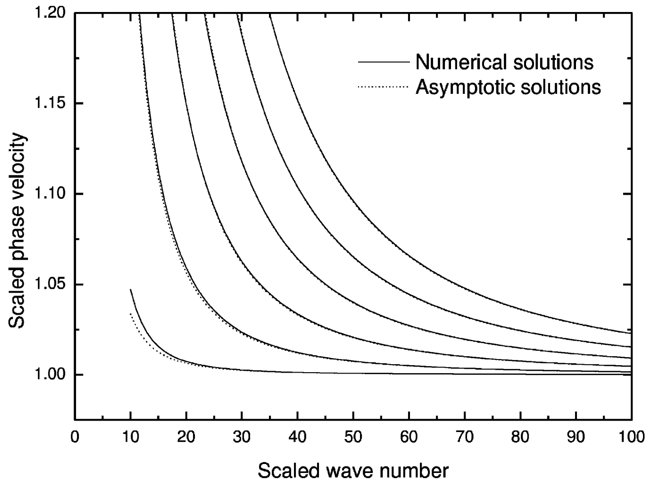

Fig. 17. Comparison of asymptotic and numerical solutions in the short wave regime; flexural case, open circuit condition.

For piezo-electric materials, the expressions of H_i and G_i, $i = 1, 2, 3$ are so complicated that it is difficult to obtain a meaningful representation of the surface wave speed equation. However, from Eqs. (97) and (98) the speed of surface wave can be obtained numerically. For PZT-4 piezo-electric ceramics, the scaled surface wave speed is 0.950168 and 0.957768 for short circuit and open circuit conditions, respectively. The effect of the two different electrical boundary conditions on the surface wave speed is small for the chosen parameters. The existence of surface waves in piezo-electric substrates is far more complicated than its elastic counterpart. So far, it has been proved that surface waves exist on a semi-infinite piezo-electric substrate with short circuit boundary conditions under virtually any conditions. However, for open circuit boundary conditions, far more stringent conditions are necessary to guarantee existence of surface waves (see Peach, 2001). Analysis of dispersion relations for a plate then provides an alternative way to compute surface wave speeds in piezo-electric media.

4. Relative orders: displacements and electric potential

In this section we shall just establish the relative orders of the displacement components and electric potential within the vicinity of the various types of cut-off frequencies which exist.

4.1. Short circuit condition

Extensional waves: In this case, the dimensionless displacements and electric potential are given by

$$\left\{\begin{matrix} \bar{U} \\ \bar{W} \\ \bar{\Phi} \end{matrix}\right\} = \sum_{l=1}^{3} \begin{pmatrix} A_l \sinh(\bar{k}q_l\xi) \\ A_l d_l \cosh(\bar{k}q_l\xi) \\ A_l f_l \cosh(\bar{k}q_l\xi) \end{pmatrix}, \tag{99}$$

where constants A_l satisfy the following equations

$$\begin{aligned} &\sum_{i=l}^{3}(q_l\bar{c}_{11} + \mathrm{i}d_l\bar{c}_{13} + \mathrm{i}\bar{e}_{31}f_l)\cosh(\bar{k}q_l)A_l = 0, \\ &\sum_{l=1}^{3}(\mathrm{i} + q_l d_l + q_l f_l \bar{e}_{15})\sinh(\bar{k}q_l)A_l = 0, \\ &\sum_{l=1}^{3} f_l \cosh(\bar{k}q_l)A_l = 0. \end{aligned} \tag{100}$$

(a) The first family of cut-off frequencies: In the vicinity of the first family of cut-off frequencies, we have $\sinh(\bar{k}q_2) \sim \mathrm{O}(1)$ and $\cosh(\bar{k}q_2) \sim \mathrm{O}(\bar{k}^2)$. Also we know $\sinh(\bar{k}q_1) \sim \mathrm{O}(\bar{k})$ and $\cosh(\bar{k}q_1) \sim \mathrm{O}(1)$. Analyzing the orders of all quantities occurring in Eq. (100), we find

$$\begin{pmatrix} a_{11}(\mathrm{O}(\bar{v}^2)) & a_{12}(\mathrm{O}(\bar{k})) & a_{13}(\mathrm{O}(\bar{v})) \\ a_{21}(\mathrm{O}(\bar{v})) & a_{22}(\mathrm{O}(1)) & a_{23}(\mathrm{O}(\bar{v}^2)) \\ a_{31}(\mathrm{O}(\bar{v}^2)) & a_{32}(\mathrm{O}(\bar{k}^3)) & a_{33}(\mathrm{O}(\bar{v})) \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix} = 0. \tag{101}$$

By comparison with Eq. (37), we conclude that $A_1 = \mathrm{O}(\bar{k})A$, $A_2 = \mathrm{O}(\bar{v}^2)A$ and $A_3 = \mathrm{O}(1)A$, where A is a disposable constant. Furthermore, we can estimate the relative order of displacements and electric potential by inserting A_l, d_l and f_l into Eq. (99), indicating that

$$\bar{U} = \mathrm{O}(\bar{v})A, \qquad \bar{W} = \mathrm{O}(1)A, \qquad \bar{\Phi} = \mathrm{O}(1)A. \tag{102}$$

In this case the leading order scaled displacement is normal to the plane of the plate and this is in fact the asymptotically leading quantity. The elastic motion is essentially thickness stretch resonance and we recall that in this case the cut-off frequencies depend only on elastic constants.

(b) The second family of cut-off frequencies: Following the procedure just employed for the first family of cut-off frequency, we obtain

$$\bar{U} = \mathrm{O}(1)A, \qquad \bar{W} = \mathrm{O}(\bar{v})A, \qquad \bar{\Phi} = \mathrm{O}(\bar{v})A. \tag{103}$$

In this case the leading order elastic displacement is in the plane of the plate and this type of motion is essentially thickness shear resonance. Moreover, the scaled in-plane displacement is the same order as the scaled electric potential and we recall that the cut-off frequencies are in this case dependent on both elastic and electrical terms.

Flexural waves: The method adopted above may now be used to estimate the relative orders of dimensionless displacements and electric potential in the flexural case. For conciseness, we only present the final results and omit details. In the vicinity of the first family of cut-off frequencies, we deduce

$$\bar{U} = \mathrm{O}(\bar{v})A, \qquad \bar{W} = \mathrm{O}(1)A, \qquad \bar{\Phi} = \mathrm{O}(1)A \tag{104}$$

In this case we recall that the cut-off frequencies depend only on elastic constants. Furthermore, we note that the leading order displacement is in-plane, indicating shear resonance, and the scaled in-plane displacement is a larger order than the scaled electric potential. For the second family of cut-off frequencies, which in this case are dependent only on both elastic and electrical terms, the relative orders of displacements and electric potential are as given by

$$\bar{U} = \mathrm{O}(1)A, \qquad \bar{W} = \mathrm{O}(\bar{v})A, \qquad \bar{\Phi} = \mathrm{O}(\bar{v})A. \tag{105}$$

In this case we have thickness stretch resonance, with the leading order scaled displacement the same order as the scaled electric potential.

4.2. Open circuit condition

For brevity, we only present the final results of relative orders and omit derivation details.

Extensional waves

(a) First family of cut-off frequencies

$$\bar{U} = \mathrm{O}(\bar{v})A, \qquad \bar{W} = \mathrm{O}(1)A, \qquad \bar{\Phi} = \mathrm{O}(\bar{v}^2)A. \tag{106}$$

We remark that in this case the cut-off frequencies were dependent on both elastic and electrical terms. Moreover, in this case the scaled electric potential is much larger than either the in-plane or normal elastic displacement.

(b) Second family of cut-off frequencies

$$\bar{U} = \mathrm{O}(1)A, \qquad \bar{W} = \mathrm{O}(\bar{v})A, \qquad \bar{\Phi} = \mathrm{O}(\bar{v})A. \tag{107}$$

This case is essentially shear resonance, with the in-plane scaled displacement the same order as the scaled electric potential, the cut-off frequencies being functions of both elastic and electric constants.

Flexural waves

(a) First family of cut-off frequencies

$$\bar{U} = \mathrm{O}(\bar{v})A, \qquad \bar{W} = \mathrm{O}(1)A, \qquad \bar{\Phi} = \mathrm{O}(1)A. \tag{108}$$

In this case the situation is similar the flexural case for the short circuit conditions. In essence, the cut-off frequencies are defined through elastic terms only and the scaled normal displacement asymptotically leads both the scaled in-plane displacement and scaled electric potential, indicating thickness stretch resonance.

(b) Second family of cut-off frequencies

$$\bar{U} = \mathrm{O}(1)A, \qquad \bar{W} = \mathrm{O}(\bar{v})A, \qquad \bar{\Phi} = \mathrm{O}(\bar{v})A. \tag{109}$$

In this, the final case, we recall that the cut-off frequencies are depend on both elastic and electrical terms and note that the in-plane scaled displacement is the leading order displacement and this is the same order as the electric potential. The associated motion is essentially thickness shear resonance.

In summary, there are three distinct patterns in the vicinity of cut-off frequencies. In the first case, $\bar{U} = \mathrm{O}(\bar{v})A$, $\bar{W} = \mathrm{O}(1)A$ and $\bar{\Phi} = \mathrm{O}(1)A$; in the second $\bar{U} = \mathrm{O}(1)A$, $\bar{W} = \mathrm{O}(\bar{v})A$ and $\bar{\Phi} = \mathrm{O}(\bar{v})A$; and in the third $\bar{U} = \mathrm{O}(\bar{v})A$, $\bar{W} = \mathrm{O}(1)A$ and $\bar{\Phi} = \mathrm{O}(\bar{v}^2)A$. We remark that in the first case, the scaled displacement in the thickness direction is much larger than scaled electric potential, however in the third pattern it is the latter which is much larger than the former. These properties suggest that an actuator may be more suitable for working within frequencies associated with first case and a sensor within those associated with the third.

5. Concluding remarks

In this paper, a comprehensive analysis of the dispersion relations for Lamb waves in a piezo-electric plate has been carried out. Numerical results, showing scaled phase speed, and scaled frequency, against wave number are presented. Guided by these numerical calculations, asymptotic expansions, giving scaled phase speed and scaled frequency as functions of wave number are derived in both the long and short wave regimes. It is found that some families of cut-off frequencies are dependent only on elastic constants, while others depend on both elastic and electrical material constants. Using these asymptotic results, the orders of mechanical displacements and electrical potential are estimated for motion within the vicinity of the cut-off frequencies. This estimation of the relative orders of displacements and electrical potential reveals that there exist three distinct deformations patterns. In one of these deformation patterns, the scaled displacement in the thickness direction is much larger than electric potential; in another the scaled electrical potential is one order larger than the largest mechanical displacement, this being in the thickness direction. These properties may be useful for the development and design of sensing and actuating devices. The asymptotic analysis has also provided the necessary theoretical basis for the derivation of asymptotically approximate models to elucidate motion near the cut-off frequencies. Such models have previously been derived for transversely isotropic elastic plates, see Kaplunov et al. (2000). Work is in progress to derive similar models in the present context and will be reported in due course.

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